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В сборнике представлены труды VIII Московской международной конференции по исследованию операций. Конференция проводится факультетом вычислительной математики и кибернетики МГУ имени М.В. Ломоносова, Федеральным исследовательским центром «Информатика и управление» Российской академии наук (ФИЦ ИУ РАН) и Российским научным обществом исследования операций (РНОИО). На конференции обсуждаются математические вопросы исследования операций в экономике, экологии, социологии, биологии, медицине, политологии, а также численные методы исследования операций.

Ключевые слова: исследование операций; математическое моделирование; методы оптимизации.

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Keywords: operations research; mathematical modeling; optimization methods.

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VIII Московская международная конференция по исследованию операций (ORM2016)

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Конференция проводится факультетом вычислительной математики и кибернетики МГУ им. М.В. Ломоносова, Федеральным исследовательским центром «Информатика и управление» РАН (ФИЦ ИУ РАН) и Российским научным общество исследования операций (РНОИО), и посвящена памяти выдающегося российского ученого, академика РАН П. С. Краснощекова. На конференции обсуждаются теоретические аспекты и различные приложения исследования операций.

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Pavel Sergeevich Krasnoschekov (06.05.1935–26.02.2016) was born in Kalach town, in Voronezhskiy region of Russia. In 1958, he graduated from Faculty of Mechanics and Mathematics of Lomonosov Moscow State University (MSU), and in 1961, he completed the aspirant (PhD) program at Steklov Institute of Mathematics. He got his candidate degree in 1964, and doctor of sciences in physics and mathematics degree in 1973. In his doctoral thesis, he studied models of large-scale military conflicts. In 1984, he was elected as a corresponding member of Academy of Sciences, and as a full member of Russian Academy of Sciences (RAS) in 1992. Since 1966, and until the end of his life, he has been working in Computing Center of RAS, as a deputy director (1989–2004), and as a chief scientific researcher (2004–2016). Since 1975, he has also been a head of Operations Research department at Lomonosov MSU. In 1981, P.S. Krasnoschekov was rewarded the State Premium for his work on theoretical foundations and practical applications of computer-aided design. These results provided a possibility for the efficient design and production of airplanes by Sukhoy plant since 1980. In 1990th, P.S. Krasnoschekov has proposed and studied a model of collective behavior with application to elections. Afterwards, he has been working on foundations of theoretical physics in the general field theory. There are more than 10 doctors and 25 candidates of sciences among his pupils. His book "Principles of Models' Design" (1983, co-authored by A. Petrov) remains a basic textbook for students at Lomonosov MSU and at Moscow Institute of Physics and Technology.

Contents

Optimization methods

Charged balls method for finding the minimum distance between two plane convex smooth curves in three-dimensional space*

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We consider the problem

$$\begin{cases} \|x-y\| \longrightarrow \min \\ x \in X \\ y \in Y \end{cases}$$

where X and Y are some plane convex smooth curves in $\mathbb{R}^3$. This problem appears in astronomy, computer graphics and many other areas. New recently described charged balls method [1], is proposed to solve the problem. This method is based on mechanic analogies [2]. The approach of passing from the original stationary problem to a nonstationary mechanical system is quite common and was used by many researchers to describe new effective optimization methods [3, 4].

It is proposed to place two oppositely charged balls onto the curves in an arbitrary points. Balls will start to move towards the equilibrium position, which obviously coincides with the solution of our problem. By

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means of Newton's second low equations of motion can be derived:

$$\begin{cases} m\ddot{\eta}_1(t) = F_1(t) + N_1(t) + R_1(t) \\ m\ddot{\eta}_2(t) = F_2(t) + N_2(t) + R_2(t) \end{cases}$$

Here m is the mass of the balls, F_1, F_2 are Coulomb forces, N_1, N_2 are normal forces, R_1, R_2 are viscous friction forces, needed to provide the tendency of η_1, η_2 (coordinates of the first and second balls correspondingly) to the equilibrium. Using numerical method for solving the obtained system of differential equations, we get the optimization algorithm for our initial problem.

Numerical experiments and animations that illustrate the work of the algorithm are presented.

References

1. Abbasov M.E. Charged balls method (in Russian). Preprint. `http://www.apmath.spbu.ru/cnsa/pdf/2015/Charged_balls.pdf` // Seminar on Constructive Nonsmooth Analysis and Nondifferentiable Optimization (CNSA & NDO), 2015.
2. Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. Numerical methods (in Russian). Moscow: Nauka, 1987.
3. Polyak B.T. Introduction to Optimization. Optimization Software, 1987.
4. Vasiliev F.P. Optimization methods (in Russian). Moscow: Factorial Press, 2002.

Pontryagin maximum principle in optimal control problems with geometric mixed constraints*

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Consider the optimal control problem

$$\begin{cases} \text{Minimize} & \varphi(p) + \int_{t_1}^{t_2} f_0(x,u,t)dt \\ \text{subject to} & \dot{x} = f(x,u,t),\ t \in T, \\ & R(x,u,t) \in C, \\ & p \in K. \end{cases} \tag{1}$$

Here, $T = [t_1, t_2]$ is the time interval (which we assume fixed, and $t_2 > t_1$), $\dot{x} = \frac{dx}{dt}$, x is state variable, which takes values in the Euclidean space $\mathbb{R}^n$, $p = (x_1, x_2)$ is the so called endpoint vector, where $x_1 = x(t_1)$, $x_2 = x(t_2)$, and $u(\cdot)$ taking values in $\mathbb{R}^m$ is the control function. The vector-function $R : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^r$ and the closed set C define the geometric mixed constraints. The control function $u(\cdot)$ is considered measurable and essentially bounded, such that, together with the arc $x(\cdot)$, satisfies the mixed constraints. The set K is closed and it defines the endpoint constraints which have to be satisfied as well. If the mixed constraints and the endpoint constraints are satisfied, then the control process (x, u) is called admissible. The control process (x^*, u^*) is called optimal, if the value of the minimizing functional at any admissible process is not less than its value at (x^*, u^*). For the classic formulation of the control problem, see [1].

The mappings in (1),

$$\begin{aligned} &\varphi : \mathbb{R}^{2n} \to \mathbb{R}^1, \\ &f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^n, \\ &f_0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^1, \text{ and} \\ &R : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^r \end{aligned}$$

satisfy the following main hypothesis. The maps f, f_0, R are continuously differentiable in (x, u) for a.a. t. On any bounded set, these maps and

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their partial derivatives in (x,u) are bounded, Lebesgue measurable in t for all (x,u), and continuous in (x,u) uniformly in t. The scalar function φ is continuously differentiable.

Everywhere in what follows, assume that problem (1) has a solution (x^*,u^*).

Consider the set-valued map

$$U(x,t) := \{u \in \mathbb{R}^m : \ R(x,u,t) \in C\}.$$

Definition 1 *A point $u \in U(x,t)$ is said to be regular provided that*

$$N_C(R(x,u,t)) \cap \ker \frac{\partial R^*}{\partial u}(x,u,t) = \{0\}. \tag{2}$$

Here, the set $N_C(y)$ designates the limiting normal cone in the sense of Mordukhovich, [2], and A^* denotes the conjugate matrix or operator A. The regularity of the point u means that the so called Robinson Constraint Qualification (RCQ) holds at u for the constraint system $R(x,u,t) \in C$, [3].

The condition (2) can be reformulated in the following way: there exists a number $\varepsilon > 0$ such that

$$\left| y\frac{\partial R}{\partial u}(x,u,t)\right| \geq \varepsilon |y|, \quad \forall\, y \in N_C(R(x,u,t)).$$

The upper bound of all such ε's is also known as *modulus of surjection* of the constraint system $M : R(x,u,t) \in C$. Let us denote the modulus of surjection to an arbitrary given constraint system $V : F(z) \in S$ at point z, by $\operatorname{sur} V(z)$.*

Then, the regularity of the point $u \in U(x,t)$ is equivalent to the relation

$$\operatorname{sur} M(x,u,t) > 0.$$

We denote by $U_{\text{reg}}(x,t)$ the subset of all regular points of $U(x,t)$. The subset of points for which $\operatorname{sur} M(x,u,t) \geq \varepsilon$ is denoted by $U^{\varepsilon}_{\text{reg}}(x,t)$. Note

*In the literature, the modulus of surjection is introduced for set-valued maps $G : X \to 2^Y$. If spaces X, and Y are finite dimensional, then

$$\operatorname{sur} G(x|y) = \inf\{|x^*| : \ x^* \in D^*G(x,y)(y^*),\ |y^*| = 1\}.$$

Here, $D^*G(x,y)$ is the limiting coderivative of G at (x,y). By definition, $\operatorname{sur} G(x|y) = \infty$ when $y \notin G(x)$. If we set $G(\cdot) := R(x,\cdot,t) - C$, then $\operatorname{sur} M(x,u,t) = \operatorname{sur} G(x,u,t|0)$.

that this set may not be closed. It is clear that

$$\begin{aligned} U^{\varepsilon}_{\rm reg}(x,t) &\subseteq U_{\rm reg}(x,t) \subseteq U(x,t) \ \ \forall\, \varepsilon>0, \text{ and} \\ U^{\alpha}_{\rm reg}(x,t) &\subseteq U^{\beta}_{\rm reg}(x,t) \ \text{ for } \alpha>\beta>0, \end{aligned}$$

and $U^0_{\rm reg}(x,t)=U(x,t)$.

The following concept corresponds to the classic approach to regularity for mixed constraints. (The so-called strong regularity.)

Definition 2 *The trajectory $x^*(t)$ is said to be regular w.r.t. the mixed constraints provided there is a number $\varepsilon_0>0$ such that*

$$U(x^*(t),t) \subseteq U^{\varepsilon_0}_{\rm reg}(x^*(t),t), \ \textit{for a.a.}\ t\in T.$$

However in what follows a weaker regularity condition will be used.

Definition 3 *The trajectory $x^*(\cdot)$ is said to be weakly regular w.r.t. the mixed constraints provided there is a number $\varepsilon_0>0$ such that*

$$u^*(t) \in U^{\varepsilon_0}_{\rm reg}(x^*(t),t) \ \ \textit{for a.a.}\ t\in T.$$

The regularity condition imposed in Definition 3 is weaker than the one from Definition 2, as it holds only locally in a small tube about $u^*(t)$, but not for all feasible points. The price to pay for this sharp drop down from the global to the local nature is the modified Weierstrass-Pontryagin maximum condition (6) that it appears in Theorem 1. See the discussion in [4] for more details and examples over the given concepts.

Along with the regularity, we also need the notion of the proper point. Let us introduce it. Let δ be a positive number and $u_0\in U(x,t)$. Along the constraint system M defining the mixed constraints in problem (1), consider the associated constraint system

$$M_{\delta,u_0}: \quad \begin{cases} R(x,u,t)\in C, \\ |u-u_0|\le\delta. \end{cases}$$

Definition 4 *A point $u_0\in U(x,t)$ is said to be proper (or, α,γ-proper) provided there exist $\alpha,\gamma>0$ such that*

$$\operatorname{sur} M_{\delta,u_0}(x,u,t)\ge\gamma \ \ \forall\, u\in U(x,t):\ |u-u_0|\le\delta, \ \ \forall\,\delta\in(0,\alpha).$$

Results of [4] suggest a large subclass of the constraint systems for which any regular point is proper. Such a subclass includes convex sets, semi-algebraic sets, or even more general than semi-algebraic type of the sets, the sets which admit the so-called Whitney stratification, i.e., satisfying the Whitney condition b).

Let us impose the following condition.

Condition P) For all $\varepsilon > 0$, $\exists\,\gamma > 0$ such that, for any measurable bounded selector $u(t)$ of the map $U^{\varepsilon}_{\mathrm{reg}}(t) := U^{\varepsilon}_{\mathrm{reg}}(x^*(t), t)$, there exists a measurable scalar function $\alpha(t)$ s.t. $u(t)$ is $\alpha(t), \gamma$-proper for a.a. t.

Condition P) may seem somewhat cumbersome, but this condition is satisfied for the above mentioned subclass of the constraint systems. This means that the result following below is valid under C convex, or semi-algebraic, or, even, when the set C admits Whitney stratification.

Following [1], we introduce the Hamilton-Pontryagin function

$$H(x, u, t, \psi, \lambda) = \big\langle \psi, f(x, u, t) \big\rangle - \lambda f_0(x, u, t).$$

Under the weak regularity condition the following theorem is true.

Theorem 1 (Maximum Principle) *Let $\varepsilon \in (0, \varepsilon_0)$. Suppose that the process (x^*, u^*) is optimal to problem (1), the arc $x^*(t)$ is weakly regular w.r.t. the mixed constraints and that Condition P) is satisfied.*

Then, there exist a number $\lambda \geq 0$, an absolutely continuous function $\psi : T \to \mathbb{R}^n$, an essentially bounded measurable function $\eta : T \to \mathbb{R}^r$, and a constant $\kappa > 0$, which all depend on ε, such that

$$\eta(t) \in \operatorname{conv} N_C(R(t)) \ \ \textit{for a.a.}\ t, \tag{3}$$

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}(t) + \eta(t)\frac{\partial R}{\partial x}(t) \ \ \textit{for a.a.}\ t, \tag{4}$$

$$(\psi(t_1), -\psi(t_2)) \in \lambda \frac{\partial \varphi}{\partial p}(p^*) + N_K(p^*), \tag{5}$$

$$\max_{u \in \operatorname{cl} U^{\varepsilon}_{\mathrm{reg}}(t)} H(u, t) = H(t) \ \ \textit{for a.a.}\ t, \tag{6}$$

$$\frac{\partial H}{\partial u}(t) - \eta(t)\frac{\partial R}{\partial u}(t) = 0 \ \ \textit{for a.a.}\ t, \tag{7}$$

$$|\eta(t)| \leq \kappa(\lambda + |\psi(t)|) \ \ \textit{for a.a.}\ t, \tag{8}$$

and

$$\lambda + |\psi(t)| > 0 \ \ \forall\, t \in T. \tag{9}$$

Here, if some of the arguments of a function or of a set-valued map are omitted, then it means that the extremal values $x^*(t)$, $u^*(t)$, $\psi(t)$, and λ are in the place of the omitted arguments.

This result covers the corresponding results from [5], where C was considered merely convex.

References

1. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko, Mathematical Theory of Optimal Processes, Moscow, Nauka, 1983.
2. B.S. Mordukhovich, Maximum principle in problems of time optimal control with nonsmooth constraints, Appl. Math. Mech., 40, 1976, pp. 960–969.
3. S.M. Robinson, Regularity and stability for convex multivalued functions, Math. Oper. Res., 1 (1976), pp. 130–143.
4. A.V. Arutyunov, D.Yu. Karamzin, F.L. Pereira, G.N. Silva. Investigation of regularity conditions in optimal control problems with geometric mixed constraints (2015) Optimization, 22 p. Article in Press.
5. A.V. Arutyunov, D.Yu. Karamzin, F.L. Pereira, Maximum Principle in Problems with Mixed Constraints under Weak Assumptions of Regularity, J. of Optimization, Volume 59, Issue 7, October 2010, pp. 1067–1083.

The algorithm for auxiliary problem in SQP-method

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Currently methods of successive quadratic programming (SQP) are among the most effective optimization methods.

Suppose that the function $f : \mathbb{R}_n \to \mathbb{R}$ and map $F : \mathbb{R}_n \to \mathbb{R}_m$ are twice differentiable on all $\mathbb{R}_n$. Consider the problem

$$f(x) \to \min, \quad x \in X = \{x \in \mathbb{R}_n \mid F(x) \le 0\}. \tag{1}$$

Let $x^k \in \mathbb{R}_n$ — the current approximation of required stationary point x^* of problem (1). The essence of the SQP-method lies in the approximation of this problem near the x^k of the quadratic programming problem types

$$\min_{x \in X_k} \left\{ g(x) = \frac{1}{2}\langle x, Hx\rangle + \langle d, x\rangle + D \right\}, X_k = \{x \in \mathbb{R}_n \mid Ax \le b\}, \tag{2}$$

where the symmetric matrix $H = f''(x^k)$ is assumed positive definite, so $g(x)$ strictly convex, $D = f(x^k) - \langle f'(x^k), x^k\rangle + f'(x^k), x^k\rangle + \frac{1}{2}\langle x^k, Hx^k\rangle$, $d = f'(x^k) - Hx^k$, $A = f'(x^k)$ — matrix of dimension $m \times n$, rank$A = m$, $m \le n$, $b = \langle f'(x^k), x^k\rangle - F(x^k) \in \mathbb{R}_m$, $X_k \neq \varnothing$.

The Lagrangian dual problem has the form

$$\min_{y\ge 0}\left\{\varphi(y) = \frac{1}{2}\langle y, Qy\rangle + \langle y, c\rangle + C\right\}, \tag{3}$$

where $Q = AH^{-1}A^\top$, $c = AH^{-1}d - b$, and constant $C = \frac{1}{2}\langle d, H^{-1}d\rangle + D$. When you made assumptions about the matrices H and A matrix Q positive definite.

First of all, note that the point $y^0 = -Q^{-1}c$ is a point unconditional minimum of the function $\varphi(y)$. Thus, if $y^0 \ge 0$, then this point — the solution of the problem (3). It is obvious also, that the solution of the problem is the point $y^* = 0$ if $c \ge 0$. Suppose that the vectors y^0 and c contain negative components.

It is known that the problem (3) can be reduced to normal form by using regular transformation of coordinates. Let the matrix U define such the conversion, i.e. $y = Uz$ and $z = U^{-1}y$. In this case transform the problem (3) takes the form

$$\min_{z\in Z}\left\{F(z) = \frac{1}{2}\sum_{i=1}^{m} z_i^2 - \langle z, p\rangle + C\right\}, \quad Z = \{z \in \mathbb{R}_m \mid Uz \ge 0\}, \tag{4}$$

where $p = -U^\top c$ and the set Z is a pointed cone in $\mathbb{R}_m$ as the rank of the matrix U is equal to m. Using, for example, the Lagrange's method full selection of square, consisting of $(m-1)$-th steps of the same type of conversion matrix coefficients Q, the quadratic form can be reduced to a canonical form. Consequently, this procedure requires $O(m^3)$ elementary operations. For reduction of quadratic form to normal form it remains to multiply the received regular matrix on diagonal that does not affect the specified computational the complexity of the procedure.

Form problems of type (4) attractive for analysis because the surfaces of level of the objective function of this problem are concentric m-dimensional sphere centered at the point p. Consequently, the solution z^* of problem (4) is a projection of the point p on a cone Z. In other words, the problem (4) an equivalent problem

$$\min_{z\in Z}\left\{\varphi(z) = \frac{1}{2}\|z - p\|^2\right\}. \tag{5}$$

To solve this problems we can use the proposed in [1] algorithm, whose computational complexity is $O(m^4)$. Hence the computational complexity of the method of solution of the problem (2). Indeed, when the reduction of the original problem to the dual problem (3) the most time consuming operation is the inverse of the matrix H, which requires $O(n^3)$ elementary operations. Reduction of quadratic form $\langle y, Qy\rangle$ to normal form associated with the implementation of $O(m^3)$ operations. Finally, the solution of the problem (5), as already noted, provides for $O(m^4)$ operations. Thus, to solve the problem (2) requires $O(n^3 + m^4)$ elementary operations. If z^* is the solution of the problem (5), the solution of the original problem (2) is a formula $x^* = H^{-1}(d - A^\top U z^*)$.

As shown in [1] the proposed algorithm is applicable to problem (2), the matrix H which is nonnegative determined. Let $f(x)$ bounded from below on X. Then the solution of the problem exists. Denote by X^* the set of its solutions.

Using the nonsingular transformation $y = V^{-1}x$ will give quadratic form to canonical form. Then the problem (2) takes the form

$$\min_{y\in Y}\left\{\psi(y) = \frac{1}{2}\langle y, \Lambda y\rangle - \langle q, y\rangle\right\}, \tag{6}$$

where Λ — diagonal matrix of size n with elements $\lambda_i, i = \overline{1,n}$ on the main diagonal, $q = dV^\top, Y = \{y \in \mathbb{R}_n \mid Gy = b\}, G = AV$. We assume that the set Y is not empty and is bounded, i.e. there exists a constant D that $\|y\| \le D$ for any $y \in Y$. Denote by Y^* the set of solutions of the problem (6) and by ψ^* the optimal value of the objective function of this problem.

Suppose the first k numbers $\lambda_i, \quad i = \overline{1,k}$ is positive and $\lambda_{k+1}, \dots, \lambda_n$ are zero. Let $\lambda^* = \min\limits_{i=\overline{1,k}} \lambda_i > 0$. Put $\Lambda_\varepsilon = \Lambda + \varepsilon I$, where I is the identity matrix of size n, and $\varepsilon : 0 < \varepsilon < \lambda^*$, and consider the problem

$$\min_{y\in Y}\left\{\psi_\varepsilon(y) = \frac{1}{2}\langle y, \Lambda_\varepsilon y\rangle - \langle q, y\rangle\right\}, \tag{7}$$

It is obvious that the problem (7) is a quadratic programming problem with positive definite quadratic form, which we will use the algorithm described above.

Let y_ε^* - solution of problem (7). In virtue of strong convexity of $\psi_\varepsilon(y)$ is the solution unique. As shown in [1] *for any* $\mu > 0$ *there is such* $\varepsilon > 0$ *that* $\psi(y_\varepsilon^*) - \psi^* < \mu$, *where* y_ε^* *— the solution of problem* (7).

Thus, it follows from the theorem of weak convergence of the algorithm when $\varepsilon \to 0$. In other words, it is possible to obtain a solution with any given accuracy in functionality, cost decision $O(n^3 + m^4)$ operations.

References

1. Bereznev V.A. A polynomial algorithm for the quadratic programming problem // Russian J. of Numerical Anal. and Math. Modelling, 2014, V.29, No 3, P.139-144.

Newton-type method for variational equilibrium problem*

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We consider the Generalized Nash Equilibrium Problem (GNEP) with two players and shared constraints:

$$\begin{array}{cc} f_1(x^1, x^2) \to \min\limits_{x^1}, & f_2(x^1, x^2) \to \min\limits_{x^2}, \\ g(x^1, x^2) \leq 0, & g(x^1, x^2) \leq 0, \end{array} \tag{1}$$

where the objective functions $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and the mapping $g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ are smooth.

A point $(\bar{x}^1, \bar{x}^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is called *generalized Nash equilibrium* if $\bar{x}^1$ is a solution of the first problem in (1) with $x^2 = \bar{x}^2$, and $\bar{x}^2$ is a solution of the second problem in (1) with $x^1 = \bar{x}^1$.

GNEPs arise in various applied and theoretical areas: economics, engineering, computer sciences, operations research, etc. This problem class has been attracting recently much attention, in particular because it turned out that the approaches and methods of modern variational analysis can be successfully applied in this context.

For each optimization problem in (1), define its Lagrangian $L_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$,

$$L_j(x^1, x^2, \mu^j) = f_j(x^1, x^2) + \langle \mu^j, g(x^1, x^2) \rangle, \quad j = 1, 2,$$

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and consider the concatenated Karush–Kuhn–Takker optimality conditions:

$$\begin{gathered}
\frac{\partial L_1}{\partial x^1}(x^1,\, x^2,\, \mu^1) = 0, \quad \frac{\partial L_2}{\partial x^2}(x^1,\, x^2,\, \mu^2) = 0, \\
\mu^1 \geq 0, \quad \langle \mu^1,\, g(x^1,\, x^2)\rangle = 0, \quad \mu^2 \geq 0, \quad \langle \mu^2,\, g(x^1,\, x^2)\rangle = 0, \\
g(x^1,\, x^2) \leq 0.
\end{gathered} \tag{2}$$

A generalized Nash equilibrium $(\bar{x}^1,\, \bar{x}^2)$ is called *variational equilibrium* if the corresponding Lagrange multipliers of two players coincide, i.e., $(\bar{x}^1,\, \bar{x}^2)$ satisfies (2) with $\bar{\mu}^1 = \bar{\mu}^2 = \bar{\mu} \in \mathbb{R}^m$. Therefore, variational equilibria are characterized by system (2), where $\mu^1 = \mu^2 = \mu$:

$$\begin{gathered}
\frac{\partial L_1}{\partial x^1}(x^1,\, x^2,\, \mu) = 0, \quad \frac{\partial L_2}{\partial x^2}(x^1,\, x^2,\, \mu) = 0, \\
\mu \geq 0, \quad g(x^1,\, x^2) \leq 0, \quad \langle \mu,\, g(x^1,\, x^2)\rangle = 0.
\end{gathered} \tag{3}$$

Variational equilibria are very important from practical point of view. For example, in various economics applications, Lagrange multipliers $\bar{\mu}^1$ and $\bar{\mu}^2$ can be interpreted as prices, and keeping them the same for both players is necessary for a solution to make practical sense.

Systems (2) and (3) can be both interpreted as mixed complementarity problems. However, unlike for (2), solutions of system (3) can naturally be isolated, and hence, can be found by methods developed for finding isolated solutions of mixed complementarity problems; see [1–3] and references therein.

In this work, we apply the algorithm from [2, 3] for finding variational equilibria. We establish global convergence properties of the algorithm, and provide the assumptions guaranteeing superlinear convergence rate.

References

1. Billups S.C. A homothopy-based algorithm for mixed complementarity problems // SIAM J. Optim. 2002. V. 12, № 3. P. 583–605.
2. Daryina A.N., Izmailov A.F., Solodov M.V. A class of active-set Newton methods for mixed complementarity problem // SIAM J. Optim. 2004. V. 15, № 2. P. 409–429.
3. Daryina A.N., Izmailov A.F., Solodov M.V. Numerical results for a globalized active-set Newton method for mixed complementarity problems // Comp. Appl. Math. 2005. V. 24. P. 293–316.

Study of a one-dimensional optimal control problem with a purely state-dependent cost

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We consider the following optimal control problem on a fixed time interval $[0,T]$:

$$J(x(t)) = \int_0^T e^{-rt} \cdot \Phi(x(t))\, dt \to \max, \tag{1}$$

$$\begin{cases} \dot{x} = f(x) + u\, g(x), & |u| \le 1, \\ x(0) = x_0, & x(T) = x_T, \end{cases} \tag{2}$$

where both the state $x(\cdot)$ and control $u(\cdot)$ variables are scalar functions. We assume that the function Φ is continuous and *unimodular*. The latter means that it has the only maximum point x^*, and moreover, it increases for $x < x^*$ and decreases for $x > x^*$. The functions f and g are differentiable, $g(x) > 0$. (Note that here we do not assume the differentiability of Φ, neither the monotonicity of f, g.) The admissible control set is $[-1,1]$. (The case of arbitrary control interval $a \le u \le b$ can be reduced to this one by a simple rescaling.) The time interval $[0,T]$ is supposed to be big enough.

We also assume that the Cauchy problem $\dot{x} = f(x) + ug(x)$, $x(0) = x_0$ has a solution on the whole interval $[0,T]$ for any admissible $u(t)$, and that some of these solutions satisfy the required terminal condition $x(T) = x_T$.

Since the problem is linear in the control and the admissible control set is convex and compact, the classical Filippov theorem [1,2] guarantees that an optimal trajectory exists. Our aim is to find it.

It follows from the properties of Φ that one should keep as close as possible to the point x^*, preferably just stay at x^*. Therefore, the character of optimal solution depends on whether the control system admits staying at the point x^* on some time interval, or not. If it does, we have $\dot{x} = 0$, then $u = -f(x^*)/g(x^*)$, which means that the solution depends on whether $u^* = -f(x^*)/g(x^*)$ is an admissible control value or not.

The problem (1)–(2) appears in a large variety of applications; for example, some models of mathematical economics can be reduced to it Usually, the only considered case is when $u^* = -f(x^*)/g(x^*)$ is admissible, even $|f(x^*)/g(x^*)| < 1$, therefore we call this case *standard.*

In a number of works (see e.g. [3 – 5], to mention just a few) this problem is solved by using the Pontryagin maximum principle (PMP). However, it can be noted that the usage of such an advanced theoretical result as PMP is excessive in this standard case, because the solution can be easily found on the base of well known facts of classical analysis by using the concept of *turnpike* and *the most rapid approach path* (MRAP). The last concept, in turn, is based on the Tchyaplygin comparison theorem for solutions of one-dimensional ODEs [6]. Some authors use also the Green theorem (e.g., [7, 8]), but this also seems redundant. Below we provide a rigorous justification of these arguments.

Moving on, we consider a modification of problem (1)–(2), when the final state $x(T)$ is free and the cost involves the so-called salvage term. In this case we give a complete solution of the problem.

All the above is related to the standard case. However, the most interesting case is the non-standard one, when $|f(x^*)/g(x^*)| > 1$. As far as we know, this case was not yet studied, though it could appear in different models as well. Here we find an optimal trajectory by using classical analysis, and then show that PMP gives the same result. The specific case of $|f(x^*)/g(x^*)| = 1$ is degenerate and not that interesting.

Thus, in some cases, problem (1)–(2) and its modifications can be solved without using of PMP. Let us emphasize that this is possible only when the state variable is one-dimensional, because in higher dimensions there are no comparison theorem for solutions of ODEs.

References

1. Filippov A.F. On Some Questions of the Optimal Regulating Theory // Vestnik Moscov. Univ, Ser. Math-Mech.,№ 2. P. 25–32.
2. Cesari L. Optimization Theory and Applications, Springer, 1983.
3. Ashmanov S.A. Introduction to Mathematical Economics. Moscow: Moscow State University, 1980 (in Russian).
4. Sethi S.P., Thompson G.L. Optimal Control Theory. Springer. 2005.
5. Geering H.P. Optimal Control with Engineering Applications. Springer, 2007.
6. Tchaplygin S.A. A new method for approximate integration of differential equations, in "S.A. Tchaplygin. Collected works". Moscow: Nauka, 1976 (in Russian).

7. Clark C.W., De Pree J.D. A Simple Linear Model for the Optimal Exploration of Renewable Resourses // Applied Mathematics and Optimization. V. 5, 1979/ P. 181–196.

Restoring the parameters of conjugated pairs of linear algebraic equation systems by a set solution

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The report observes the theorem of recovering the parameters of a conjugated pair of linear algebraic equation systems by a set solution using an interval criterion. Tasks in similar statements are considered in articles [5], [6].

Theorem. *The* $A \in \mathbf{R}^{m\times n}$ *family of matrices and the* $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, *families of vectors that guarantee that the set* $\bar{x} \in \mathbf{R}^n$ *and* $\bar{u} \in \mathbf{R}^m$ *vectors belong to the*

$$\begin{cases} Ax = b, \\ u^\top A = c^\top, \end{cases} \tag{1}$$

set of solutions of a conjugated pair of systems of linear algebraic equations, and at the same time, $\|A\| \le \alpha$, $\|b\| \le \beta$, $\|c\| \le \gamma$, *where* $\alpha > 0$, $\beta > 0$, $\gamma > 0$ *can be constructed using*

$$b = \lambda \frac{\bar{u}}{\bar{u}^\top \bar{u}} + \lambda \left(I_m - \frac{\bar{u}\bar{u}^\top}{\bar{u}^\top \bar{u}} \right) \Delta b, \tag{2}$$

$$c = \lambda \frac{\bar{x}}{\bar{x}^\top \bar{x}} + \lambda \left(I_n - \frac{\bar{x}\bar{x}^\top}{\bar{x}^\top \bar{x}} \right) \Delta c, \tag{3}$$

$$A = \frac{1}{\lambda} b c^\top, \tag{4}$$

formulas, where $\|\cdot\|$ *stands for, depending on the content, the Euclidean matrix or vector norm, the scalar parameter* λ *is calculated using the*

$$\lambda \le \bar{\lambda} = \min\left(\frac{\alpha}{\bar{\alpha}}, \frac{\beta}{\bar{\beta}}, \frac{\gamma}{\bar{\gamma}} \right), \tag{5}$$

rule,

$$\bar{\beta} = \sqrt{\frac{1}{\bar{u}^\top \bar{u}} + \Delta b^\top \left(I_m - \frac{\bar{u}\bar{u}^\top}{\bar{u}^\top \bar{u}} \right) \Delta b}, \tag{6}$$

$$\bar{\gamma} = \sqrt{\frac{1}{\bar{x}^\top \bar{x}} + \Delta c^\top \left(I_n - \frac{\bar{x}\bar{x}^\top}{\bar{x}^\top \bar{x}} \right) \Delta c}, \tag{7}$$

$$\bar{\alpha} = \bar{\beta} \cdot \bar{\gamma}, \tag{8}$$

$\Delta b \in \mathbf{R}^m$, $\Delta c \in \mathbf{R}^n$ *are random vectors,* I_m*,* I_n *are singular matrices of size* m *and* n*, accordingly.*

At the same time

$$\|A\| = \lambda \cdot \bar{\alpha}, \tag{9}$$

$$\|b\| = \lambda \cdot \bar{\beta}, \tag{10}$$

$$\|c\| = \lambda \cdot \bar{\gamma}. \tag{11}$$

On the basis of the theorem 1 it is possible to develop methods of the solution of the tasks described in articles [1]–[4].

The report ends with a numerical experiment with a model example. Initial parameters of the task (1):

$$x = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad u = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 10 \end{bmatrix}.$$

$$\alpha = 2, \quad \beta = 1, \quad \gamma = 0.5.$$

We will set parameters Δb, Δc as follows

$$\Delta b = \begin{bmatrix} 0.850679 \\ 0.558565 \\ 0.901774 \\ 0.419518 \end{bmatrix}, \quad \Delta c = \begin{bmatrix} 0.358128 \\ 0.488988 \\ 0.255962 \\ 0.929169 \\ 0.466757 \end{bmatrix}.$$

According to (5)–(8)

$$\bar{\beta} = 1.234372, \quad \bar{\gamma} = 0.856068, \quad \bar{\alpha} = 1.056707,$$

$$\lambda = 0.584066.$$

Further, from the (2)–(4) we obtain

$$A = \begin{bmatrix} 0.073855 & -0.028283 & 0.109244 & 0.317574 & 0.120217 \\ 0.050392 & -0.019298 & 0.074539 & 0.216686 & 0.082026 \\ 0.085080 & -0.032581 & 0.125848 & 0.365843 & 0.138489 \\ -0.018211 & 0.006974 & -0.026938 & -0.078308 & -0.029644 \end{bmatrix},$$

$$b = \begin{bmatrix} 0.426799 \\ 0.291212 \\ 0.491668 \\ -0.105241 \end{bmatrix}, \quad c = \begin{bmatrix} 0.101069 \\ -0.038704 \\ 0.149499 \\ 0.434594 \\ 0.164515 \end{bmatrix}.$$

Check shows what according to (9)–(11) is carried out

$$\|A\| = \lambda \cdot \bar{\alpha} = 0.617186 < \alpha = 2,$$

$$\|b\| = \lambda \cdot \bar{\beta} = 0.720954 < \beta = 1,$$

$$\|c\| = \lambda \cdot \bar{\gamma} = 0.5 = \gamma.$$

The equations (1) are solvable.

References

1. Erokhin V.I. Matrix correction of a dual pair of improper linear programming problems // Computational Mathematics and Mathematical Physics. 2007. V. 47. № 4. P. 564-578.
2. Erokhin V.I., Krasnikov A.S., Khvostov M.N. Matrix corrections minimal with respect to the euclidean norm for linear programming problems // Automation and Remote Control. 2012. Рў. 73. № 2. P 219-231.
3. Erokhin V.I., Krasnikov A.S. Matrix correction of a dual pair of improper linear programming problems with a block structure // Computational Mathematics and Mathematical Physics. 2008. Рў. 48. № 1. P 76-84.
4. Erokhin V.I., Laptev A.Yu., Lisitsyn N.V. Reconciliation of material balance of a large petroleum refinery in conditions of incomplete data // Journal of Computer and Systems Sciences International. 2010. V. 49. № 2. P. 295-305.
5. Gorelik V.A., Erokhin V.I., Pechenkin R.V. Minimax matrix correction of inconsistent systems of linear algebraic equations with block matrices of coefficients // Journal of Computer and Systems Sciences International. 2006. V. 45. № 5. P. 727-737.

6. Volkov V.V., Erokhin V.I. Tikhonov solutions of approximately given systems of linear algebraic equations under finite perturbations of their matrices // Computational Mathematics and Mathematical Physics. 2010. V. 50. № 4. P. 589–605.

Methods and software infrastructure for high performance optimization*

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We consider the following optimization problem

$$f(x) \to \min, \text{ s.t. } g(x) \leq 0, \tag{1}$$

where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ are continuous mappings. Finding the exact minimum f_* is usually impossible. Thus the goal is to find ε, δ-*solution* defined as follows: $x \in \mathbb{R}^n$, $g_i(x) \leq \delta, \ i = 1, \dots, m$, $f(x) \leq f_* + \varepsilon$.

Te Non-uniform Covering Method proposed in [1] is able to find ε, δ-*solution* in a finite number of steps. For realistic problems the number of steps can be quite large. Numerous techniques to reduce the number of steps have been proposed so far [2,3].

To support a variety of covering procedures we developed an object-oriented flexible and extensible software infrastructure. In this framework one can easily implement new methods to construct coverages and combine them.The core class of this software environment is `Cover`. Covers are constructed by *cover factories* inherited from the abstract class
`CoverFactory`. At the moment factories relying on comparing lower and upper bounds on an objective function, first and second order optimality conditions are implemented.

Though advanced covering techniques significantly increases the performance of the method for many practical problems the amount of required resources is beyond the capacity of a single CPU computer. For such problems the use of parallel and distributed computing is inevitable.

We created a software infrastructure that supports parallel (distributed memory) tree search scheme. The approach implemented by our tool separates the problem dependent part from the parallel

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implementation and from the logic of parallelization. The main issue in parallel tree search is load balancing. Since the structure of the tree is not known in advance the static distribution is usually not efficient. To overcome this problem parallel solvers use dynamic load balancing to distribute the computational load among processors.

In our tool special components called *schedulers* are used for managing parallel resolution process. A scheduler interact via a strictly defined interface with a solver and a parallel platform. It communicates with the parallel platform by means of special commands such as:

- send N subproblems to the process P;
- send incumbent to the process P;
- send control command to the process P;
- recieve information (subproblems, incumbent or control command) from the process P.

It is worth noting that this set of commands is problem-independent. And thus it is possible to separate the logic of the parallel processing management and the problem specific implementation of those commands. Such separation is important for several reasons. First, it saves efforts when implementing new problem because only the problem-specific part has to be implemented and the scheduler is reused. Second, common part can be a subject for a separate study. For instance it is possible to compare different load balancing strategies on a simulator or check the correctness of the parallel algorithm, e.g. identify possible deadlocks.

The simulator transparently substitutes the real parallel system and the real solver. Thus we can conveniently evaluate the performance of scheduling algorithms incorporated in our tool. Besides the simulator we also developed a graphical front-end that visualizes the processors load and communication among processors.

References

1. Evtushenko Y. G. Numerical methods for finding global extrema (case of a non-uniform mesh) //USSR Computational Mathematics and Mathematical Physics. 1971. V. 11. No. 6. P. 38–54.

2. Evtushenko Y., Posypkin M. A deterministic approach to global box-constrained optimization //Optimization Letters. 2013. V. 7. No. 4. P. 819–829.
3. Evtushenko Y. G., Posypkin M. A. Versions of the method of nonuniform coverings for global optimization of mixed integer nonlinear problems // Doklady Mathematics. MAIK Nauka/Interperiodica, 2011. V. 83. No. 2. P. 268–271.

The Minkowski difference of sets with the constraint structure

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The analytical expression of the Minkowski difference of sets has its own independent significance in many areas of mathematical sciences. In [1]–[4], we used the Minkowski difference for investigation of the sets separation problems. In this thesis, we shall demonstrate that the Minkowski difference is a useful tool for solving of the variational inequalities interconnected with the linear separation problems.

In a wide range of applications of variational inequalities, the set Φ is determined by a system of inequalities:

$$\Phi = \{x \in X : f_i(x) \leq b_i,\, i \in I\},\;\; I = \{1, 2, \cdots, m\}, \tag{1}$$

where $f_i(x)$, $i \in I$ are arbitrary real-scaled quasi-convex functions which are defined on a convex set $X \subseteq \mathbb{R}^n$.

The basic impediment to making use of operation of Minkowski difference are problems related to its implementation for different formulations of sets.

Let us recall that, in [2], we proved that the set $\Phi - \Psi$ coincides with the convex hull of the vectors $z_k - p_l$, $k \in K$, $l \in L$ if $\Phi = co\{z_k\}_{k\in K}$, $\Psi = co\{p_l\}_{l\in L}$, $K = \{1, 2, \cdots, r\}$, $L = \{1, 2, \cdots, s\}$.

Next, we presented in [4] the analytical expression of the Minkowski difference of two sets Φ and Ψ, when Φ is given by (1), and Ψ is an arbitrarily defined set.

Let be given an arbitrary set $\Psi \subseteq \mathbb{R}^n$, the set Φ be defined by (1), $X = \mathbb{R}^n$, then $\Phi - \Psi = \Phi_1$, where

$$\Phi_1 = \{x \in \mathbb{R}^n : f_i(x + y) \leq b_i,\, i \in I,\, y \in \Psi\},$$

$$I = \{1, 2, \cdots, m\},\;\; \Phi - \Psi = \{z \in \mathbb{R}^n : z = x - y,\, x \in \Phi,\, y \in \Psi\}.$$

From above, we observe that it really did not matter how the set Ψ was defined analytically or by some other way. For example, Ψ may be defined in a similar way as the set Φ:

$$\Psi := \{x \in \mathbb{R}^n : g_j(x) \le d_j,\, j \in J\},\, J = \{1, 2, \cdots, k\}.$$

It is quite clear that if the set Φ is prescribed by strict constraints, then Φ_1 should be defined by the system of the strict inequalities, too.

In particular, the set Ψ can contain a single point. For this case, we consider below some examples.

1. Let $\Phi \ne \emptyset$ be defined by (1), $p \in \mathbb{R}^n$,

$$\Phi_1 = \{x \in \mathbb{R}^n : f_i(x+p) \le b_i,\, i \in I\},\;\; I = \{1, 2, \cdots, m\},$$

then $\Phi - p = \Phi_1$, where $\Phi - p = \{z \in \mathbb{R}^n : z = x - p,\, x \in \Phi\}$.

2. If $p \in \mathbb{R}^n, i \in I,\;\; I = \{1, 2, \cdots, m\},\;\; \Phi \ne \emptyset$,

$$\Phi = \{x \in \mathbb{R}^n : \langle a_i,\, x\rangle \le b_i,\, a_i \in \mathbb{R}^n,\, b_i \in \mathbb{R}^1\}, \tag{2}$$

then $\Phi - p = \{x \in \mathbb{R}^n : \langle a_i,\, x\rangle \le \tilde{b_i},\, \tilde{b_i} = b_i - \langle a_i,\, p\rangle,\, i \in I\}$.

3. If $\Phi = \{x \in \mathbb{R}^n : l \le x \le u,\;\; l,\, u \in \mathbb{R}^n\},\;\; \Phi \ne \emptyset$, then

$$\Phi - p = \{x \in \mathbb{R}^n : l - p \le x \le u - p\}.$$

4. If $\Phi = \{x \in \mathbb{R}^n : \|x - o\|^2 \le r^2,\, o \in \mathbb{R}^n,\, r \in \mathbb{R}^1_+\}$, then

$$\Phi - p = \{x \in \mathbb{R}^n : \|x - \bar{o}\|^2 \le r^2,\, \bar{o} = o - p\}.$$

5. If $p = (p^1, \cdots, p^n), \Phi = \mathbb{R}^n_+ = \{x = (x^1, \cdots, x^n) : x^j \ge 0, j = \overline{1,\, n}\}$, then

$$\Phi - p = \{x = (x^1,\, \cdots,\, x^n) : x^j \ge -p^j,\, j = \overline{1,\, n}\}.$$

Let the set Φ be given by (2) and Ψ be described as follows

$$\Psi = \{y \in \mathbb{R}^n : \langle c_j,\, y\rangle \le d_j,\, c_j \in \mathbb{R}^n,\, d_j \in \mathbb{R}^1\},\;\; J = \{1, 2, \cdots, k\},$$

then

$$\Phi - \Psi = \{x \in \mathbb{R}^n : \langle a_i,\, x\rangle + \langle a_i,\, y\rangle \le b_i, y \in \Psi\}.$$

Let be given the arbitrary nonempty sets $\Phi, \Psi \subset \mathbb{R}^n$. If the variational inequality consists in determining a vector $c \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that

$$\langle c, x - y - c\rangle \ge 0 \quad x \in \Phi,\, y \in \Psi, \tag{3}$$

then (3) can be solved by multiple sequential projections to each region. Instead of this method for solving of (3), we can make use the single projection of the origin of $\mathbb{R}^{2n}$ onto the Minkowski difference of the sets Φ and Ψ.

Naturally, if the sets Φ and Ψ are nonempty convex and closed, and at least one of them is bounded, then $\Phi - \Psi$ is a convex and closed set. Consequently, the operation of projection onto $\Phi - \Psi$ is well defined.

Let $\mathbf{P}_{\Phi-\Psi}(\mathbf{0})$ stand for the projection of the origin onto $\Phi - \Psi$. If $\mathbf{P}_{\Phi-\Psi}(\mathbf{0}) \neq \mathbf{0}$, then it obviously holds that $\mathbf{0} \notin \Phi - \Psi$. Therefore, there exist the points $\bar{x} \in \Phi$ and $\bar{y} \in \Psi$ such that $\bar{x} - \bar{y} = \mathbf{P}_{\Phi-\Psi}(\mathbf{0})$, $\bar{x} \neq \bar{y}$. These closest points of Φ and Ψ can be found by solving of the following system:

$$\langle c, x - \bar{x} \rangle \geq 0, \quad x \in \Phi, \tag{4}$$

$$\langle c, \bar{y} - y \rangle \geq 0, \quad y \in \Psi, \tag{5}$$

where $c = \mathbf{P}_{\Phi-\Psi}(\mathbf{0})$.

Under assumption that both sets Φ and Ψ are bounded, the continuous function $\langle c, x \rangle$ attains its maximum and minimum values on the compact sets Φ and Ψ. As a consequence, the points $\bar{x}$ and $\bar{y}$ satisfying to (4)–(5) can be found by solving the following problems, respectively:

$$\min_{x \in \Phi} \langle c, x \rangle,$$

$$\max_{y \in \Psi} \langle c, y \rangle.$$

Let us notice that the vector $(\bar{x}, \bar{y})$, $\bar{x} \in \Phi, \bar{y} \in \Psi$ satisfying to (4)–(5) is the solution of the following problem:

$$\min_{x \in \Phi,\, y \in \Psi} \|x - y\|^2.$$

So, the problem of determining the distance between the sets Φ and Ψ can be solved by reduction to the next problem:

$$\min_{z \in \Phi - \Psi} \|z\|^2.$$

Consequently, the distance between the sets Φ and Ψ is equal to $\|\mathbf{P}_{\Phi-\Psi}(\mathbf{0})\|$.

References

1. Gabidullina Z.R. A Theorem on Separability of a Convex Polyhedron from Zero point of the Space and Its Applications in

Optimization // Izvestiya VUZ. Matematika. 2006. № 12. P. 21–26. (Engl.trasl. Russian Mathematics (Iz.VUZ). 2006. V. 50, № 12. P. 18–23.
2. Gabidullina Z.R. A Theorem on Strict Separability of Convex Polyhedra and Its Applications in Optimization // Journal of Optimization Theory and Applications. 2011. V. 148, № 3. P. 550–570
3. Gabidullina Z.R. A Linear Separability Criterion for Sets of Euclidean Space // Journal of Optimization Theory and Applications. 2013. V. 158, № 1. P. 145–171
4. Gabidullina Z.R. Necessary and Sufficient Conditions for Emptiness of the Cones of Generalized Support Vectors // Optimization Letters. 2015. V. 9, № 4. P.693-729, Springer Berlin Heidelberg, Available at http://link.springer.com/article/10.1007/s11590-014-0771-5

Properties of the shortest curve in a compound domain*

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A closed state domain given by constraints of the form $g_1(x) = 0$ and $g_2(x) \leq 0$ is considered, where $x \in R^n$ and g_1 and g_2 are given functions ranging in R^{k_1} and R^{k_2}, respectively. Such a state domain will be called a compound domain in what follows. In addition, throughout the following we assume that the vectors $\frac{\partial g_1^i}{\partial x}(x),\ i = 1, ..., k_1$, and $\frac{\partial g_2^j}{\partial x}(x),\ j \in J(x)$, are linearly independent for every x. Here $J(x) := \{j : g_2^j(x) = 0\}$.

Some properties of the shortest curve in a compound domain are studied. The equation of the shortest curve is derived. It is important to note the following. It might seem that the equation of the shortest curve in the presence of inequalities is a trivial consequence of the optimality principle. Indeed, any part of the shortest curve is a shortest curve itself; then, by considering its separate parts lying on the boundary of the

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domain $x : g_2(x) \leq 0$ and inside it (assume that $k_2 = 1$), we obtain the desired result. This method applies if these parts lie entirely on the boundary or inside the domain. However, such a part of the shortest curve lying entirely on the domain boundary does not necessarily exist, while the set of points of exit of the shortest curve to the boundary can be, for example, a Cantor set of positive measure. Let us give an example.

Let $C \subset [0,1]$ be a Cantor set of positive measure. Since C is closed, it follows from the Whitney theorem that there exists a nonpositive function $f : [0,1] \to R$ such that $f^{-1}(\{0\}) = C$. Take $n = 2$ and $g_2(x) = f(x_1) - x_2$ and assume that equality constraints are absent. Obviously, the shortest curve joining the points $(0,0)$ and $(1,0)$ is defined by the formulas $x_1(t) = t, x_2(t) = 0, t \in [0,1]$. One can readily see that the set $C \times \{0\}$ lies on the boundary of the domain, and the set $([0,1] \setminus C) \times \{0\}$ lies in its interior.

Note also that we should study the class of functions to which the shortest curve belongs. Obviously, in the presence of inequalities it does not belong to the class $C_2([0,1])$, in contrast to the geodesics. One can readily construct a related example.

Consider the compound domain

$$M := \{x \in R^n : g_1(x) = 0, \; g_2(x) \leq 0\},$$

and let A and B be two given points in M, $A \neq B$. Consider a smooth curve $x(t) : [0,1] \to M$ lying entirely in M and joining the points A and B; i.e. $x(0) = A$ and $x(1) = B$. (We assume that M is a connected domain; then, by virtue of the above-imposed regularity conditions, there always exists such a curve.) The shortest curve in M is defined as a continuously differentiable regular curve $x_*(t)$ with the natural parametrization that has the minimum length of all smooth curves $x(t)$ that lie in M and connect the points A and B.

Consider the control problem

$$\begin{aligned} &\frac{1}{2}\int_0^1 |u(t)|^2 dt \to \min, \; \dot{x} = u, \\ &g_1(x) = 0, \; g_2(x) \leq 0, \qquad\qquad (1) \\ &u \in R^n, \; x(0) = A, \; x(1) = B. \end{aligned}$$

Lemma 1 There exists a shortest curve $x_*(t)$ connecting the points A and B. Every shortest curve is a solution of problem (1). The converse is also true: each solution of problem (1) is a shortest curve.

Lemma 2 The shortest curve $x_*(t)$ is a function of the class $W_{2,\infty}([0,1])$. In the case without inequality constraints, it belongs to the class $C_2([0,1])$.

Lemma 3 The shortest curve $x_*(t)$ satisfies the equation

$$\ddot{x} = -g_x'^*(x)P^*(x)[P(x)g_x'(x)g_x'^*(x)P^*(x)]^{-1}P(x)g_{xx}''[\dot{x},\dot{x}] \quad (2)$$

almost everywhere on [0, 1].

Above, where P(x) is the $(k_1+k_2)\times(k_1+|J(x)|)$ matrix that takes each vector $y = (y_1, y_2, ..., y_{k_1}, y_{k_1+1}, y_{k_1+2}, ..., y_{k_1+k_2})$ to the vector $\tilde{y} = (y_1, y_2, ..., y_{k_1}, y_{k_1+j_1}, y_{k_1+j_2}, ..., y_{k_1+j_k})$, where $j_1, j_2, ..., j_k$ are the indices forming the set $J(x)$ and $g = (g_1, g_2)$.

Remark 1

Along with Eq. (2), we have the equation of the shortest curve in the simpler geometric form

$$\ddot{x} \in N_M(x).$$

Remark 2

If equality state constraints are absent, then the problem on the shortest curve for a complex- shaped domain is also referred to as the obstacle bypass problem [1, p. 66]. The possibility to derive the equation of the shortest curve from the Pontryagin maximum principle was pointed out by Gamkrelidze [2; 3, p. 347].

The proofs of these results can be found in [4]. The proofs use the theory developed in [5].

References

1. Arnol'd, V.I. Teoriya katastrof (Catastrophe Theory). Moscow: Nauka, 1990.
2. Gamkrelidze R.V. Time-Optimal Processes with Bounded State Coordinates // Dokl. Akad. Nauk SSSR. 1959. V. 125, N. 3. P. 475–478.
3. Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. Matematicheskaya teoriya optimal'nykh protsessov (Mathematical Theory of Optimal Processes). Moscow: Nauka, 1983.
4. Davydova A.V., Karamzin D.Yu. On some properties of the shortest curve in a compound domain // Differential Equations. 2015. V. 51, N. 12. P. 1626–1636.
5. Arutyunov A.V., Karamzin D.Yu. Non-Degenerate Necessary Optimality Conditions for the Optimal Control Problem with

Equality-Type State Constraints // J. of Global Optimization. 2015.

Fractional programming via D.C. optimization*

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The paper addresses the development of efficient methods for fractional programming problems [1] as follows

$$(\mathcal{P}) \qquad f(x) := \sum_{i=1}^{m} \frac{\psi_i(x)}{\phi_i(x)} \downarrow \min_x, \quad x \in S,$$

where $\phi_i(x) > 0, \;\; \psi_i(x) > 0, \;\; i = 1, \ldots, m, \;\; \forall x \in S.$

This is a nonconvex problem with multiple local extremum which belongs to a class of global optimization.

Together with problem $(\mathcal{P})$ we will also consider the following parametric optimization problem

$$(\mathcal{P}_\alpha) \qquad \Phi_\alpha(x) \stackrel{\triangle}{=} \Phi(x, \alpha) := \sum_{i=1}^{m} [\psi_i(x) - \alpha_i \phi_i(x)] \downarrow \min_x, \quad x \in S,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)^\top \in I\!R^m$ is the vectorial parameter.

Let us introduce then the optimal value function $\mathcal{V}(\alpha)$ of Problem $(\mathcal{P}_\alpha)$ as follows

$$\mathcal{V}(\alpha) := \inf_x \{\Phi_\alpha(x) \mid x \in S\}.$$

In addition, suppose that the following assumptions are fulfilled:

(a) $\mathcal{V}(\alpha) > -\infty \;\; \forall \alpha \in \mathcal{K}$, where $\mathcal{K}$ is a convex set from $I\!R^m$;

$(\mathcal{H}_1)$ $\quad (b)$ $\forall \alpha \in \mathcal{K} \subset I\!R^m$ there exists a solution $z = z(\alpha)$ to Problem $(\mathcal{P}_\alpha)$, i.e. $\mathcal{V}(\alpha) = \sum\limits_{i=1}^{m} [\psi_i(z) - \alpha_i \phi_i(z)]$.

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Then it takes place the reduction (equivalence) theorem for the fractional programming problem with d.c. functions and the solution of the equation $\mathcal{V}(\alpha) = 0$ with the vector variable $\alpha = (\alpha_1, \dots, \alpha_m)^T$ satisfying the following nonnegativity assumption

$$(\mathcal{H}(\alpha)) \qquad \psi_i(x) - \alpha_i \phi_i(x) \geq 0 \ \ \forall x \in S, \ \ i = 1, \dots, m.$$

Theorem. *Suppose that in Problem $(\mathcal{P})$ the assumptions $(\mathcal{H}_1)$ are fulfilled. In addition, let there exist a vector*

$$\alpha_0 = (\alpha_{01}, \dots, \alpha_{0m})^\top \in \mathcal{K} \subset I\!R^m$$

for which the assumption $(\mathcal{H}(\alpha_0))$ is satisfied.

Besides, suppose that in Problem $(\mathcal{P}_{\alpha_0})$ the following equality holds

$$\mathcal{V}(\alpha_0) \stackrel{\triangle}{=} \min_x \left\{ \sum_{i=1}^m [\psi_i(x) - \alpha_{0i}\phi_i(x)] \ \mid \ x \in S \right\} = 0.$$

Then any solution $z = z(\alpha_0)$ to Problem $(\mathcal{P}_{\alpha_0})$ turns out to be a solution to Problem $(\mathcal{P})$, so that $z \in Sol(\mathcal{P}_{\alpha_0}) \subset Sol(\mathcal{P})$.

This theorem opens the door to a justified use of the Dinkelbach's approach for solving fractional programming problems with the goal function presented by a sum of fractions all given by d.c. functions.

Therefore, instead of solving Problem $(\mathcal{P})$ we propose to combine a solving Problem $(\mathcal{P}_\alpha)$ with a search with respect to parameter $(\alpha \in I\!R^m_+)$ in order to find a vector $(\alpha_0 \in I\!R^m_+)$ such that

$$\mathcal{V}(\alpha_0) = \mathcal{V}(\mathcal{P}_{\alpha_0}) = 0.$$

In this situation for every $(\alpha \in I\!R^m_+)$ we must be able to find a global solution to Problem $(\mathcal{P}_\alpha)$ and we can do it using the global search theory for d.c. optimization problems [2].

Besides, we combine the developed method with another approach to the fractional programming which implies the reduction to the optimization problem of the form [3]

$$(\mathcal{P}_1) \qquad \left\{ \begin{array}{c} \sum\limits_{i=1}^m \alpha_i \downarrow \min\limits_{(x,\alpha)}, \ \ x \in S, \\ \\ \psi_i(x) - \alpha_i \phi_i(x) \leq 0, \ \ i = 1, \dots, m, \end{array} \right.$$

where $\phi_i(x) > 0,\ \ \psi_i(x) > 0,\ \ i = 1, \ldots, m,\ \ \forall x \in S.$

Furthermore, using the global search theory for problems with d.c. constraints [4]–[6], we proposed the global search method for solving the fractional programming problem $(\mathcal{P})$ via the combination of methods for problems $(\mathcal{P}_\alpha)$ and $(\mathcal{P}_1)$.

Finally, rather large field of computational simulation testings have been carried out for some special test functions formed by linear and/or convex quadratic functions.

First, the computational experiments have been performed on the small dimension's examples from [7]. Afterwords, the approach has been tested on specially designed test problems up to dimension $n = m = 100$. At the end, the test problems of dimension up to $n = m = 200$ designed with the help of [8] have been also solved by the developed algorithms.

After analysis the results of computational simulations look rather promising and competitive.

References

1. Frenk J. B. G., Schaible S. Fractional programming // Handbook of Generalized Convexity and Generalized Monotonicity (ed. by N. Hadjisavvas, S. Komlosi, S. Schaible), Series Nonconvex Optimization and Its Applications, V. 76, Springer, 2002. P. 335–386.
2. Strekalovsky A.S. Elements of nonconvex optimization. Novosibirsk: Nauka, 2003 (in Russian).
3. Dur M., Horst R., Thoai N.V. Solving sum-of-ratios fractional programs using efficient points // Optimization. 2001. V. 49. P. 447–466.
4. Strekalovsky A.S. Minimizing sequences in problems with d.c. constraints // Computational Mathematics and Mathematical Physics. 2005. V. 45(3). P. 435–447.
5. Strekalovsky A.S. On local search in d.c. optimization problems // Applied Mathematics and Computation. 2015. V. 255. P. 73–83.
6. Gruzdeva T.V., Strekalovsky A.S. Local search in problems with nonconvex constraints // Computational Mathematics and Mathematical Physics. 2007. V. 47(3). P. 381–396.
7. Ma B., Geng L., Yin J., Fan L. An effective algorithm for globally solving a class of linear fractional programming problem // Journal of software. 2013. V. 8(1). P. 118–125.
8. Jong Y.-C. An eficient global optimization algorithm for nonlinear sum-of-ratios problem // Repository of e-prints about optimization

and related topics. 2012. `http://www.optimization-online.org/DB_FILE/2012/08/3586.pdf`

When the solutions of complementarity problems are monotone with respect to parameters*

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In many applied problems (such as, e.g., elasto-hydrodynamic lubrication problem, some economic equilibrium problems, *etc.*), one of the important question is if certain complementarity problem's solution is monotone with respect to parameters. Our paper investigates this question and provides several sufficient conditions that guarantee such a monotonicity of the solutions to linear and nonlinear complementarity problems with parameters. In the majority of cases, it is required that the principal mapping of the complementarity problem be monotone by decision variables and, vice versa, antitone with respect to parameters.

The nonlinear complementarity problem (*CP*) is well-known and can be stated as follows: Given a continuous mapping $f : R^n_+ \to R^n$, find an n-vector $z \in R^n$ such that

$$z \geq 0, f(z) \geq 0, \quad \text{and} \quad z^T f(z) = 0. \tag{1}$$

A parametric version of the linear complementarity problem (*i.e.*, when f is affine) was formulated by Maier [1]. The problem of monotonicity of solutions in the parametric linear complementarity problem (PLCP) was also studied by Cottle [2] who assumed the matrix M of the parametrized mapping $f(z;t) = Mz+q+tp$ either to be *positive semi-definite* (PSD), or else to have *positive principal minors* (PM).

The results of Cottle were later generalized by Megiddo [3] who went even further in [4] and examined the general *nonlinear* parametric

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complementarity problem (NPCP) in the form: Given a continuous mapping $g : R_+^{n+1} \to R^n$, solve a family $\{g(\cdot; t) : t \geq 0\}$ of non-parametric *CP*s.

Both Maier [1] and Cottle [2] claimed that the monotonicity property in linear parametric complementarity problems (LPCP) is often desired in the context of elastoplastic structures. Cottle also suggested that a generalization of his results "would find applications in structural mechanics as well as economic equilibrium theory". All that was later confirmed in numerous papers (*see*, e.g., Kostreva [5], Ferris and Pang [6], to mention only few).

In contrast to the original problem's formulation by Maier, Cottle, and Megiddo, who tried to find not only sufficient but also *necessary* conditions of the monotonicity of the solutions of the corresponding parametric complementarity problems with respect to the parameters, we are to consider and examine a bit simpler task. Namely, we are interested in finding only *sufficient* conditions of the latter monotonicity, and because of that, we study a more general problem than that examined in [1–6].

Now consider a nonlinear complementarity problem with parameters: Given a parameter vector $u = (u_1, u_2, ..., u_m) \in R^m$, find a point $x \in R^n$ such that

$$x \geq 0, \ Ax + Bu + \varphi(x, u) \geq 0, \quad \text{and}$$

$$x^T(Ax + Bu + \varphi(x, u)) = 0; \tag{2}$$

here A, B are given $n \times n$ and $n \times m$ real matrices, and $\varphi : R^n \times R^m \to R^n$ is a nonlinear function.

In order not to restrict our research to the case of equal numbers of decision variables and parameters, we will use not the concept of monotonicity defined by the inner product of the vector-function and the vector of parameters, but the component-wise monotonicity notion (*cf.*, e.g., [7]) given below.

Definition 1. A mapping $f : R^n \to R^m$ is called *monotone* [*antitone*] if $x_1 \geq x_2$ implies $f(x_1) \geq f(x_2)$ $[f(x_1) \leq f(x_2)]$. (We say that $a \geq b$ if $a_i \geq b_i, \ i = 1, ..., n$, i.e., the partial order relation in vector spaces is involved).

Now the following result can be established. The definition and important properties of M-matrices can be found, e.g., in [8].

Theorem 1. *Let A be a positive definite M-matrix, B a non-positive one, and $\varphi(x, u)$ a differentiable function monotone by x and antitone* *with respect to u. Moreover, suppose* $\varphi'_x = \varphi'_x(x,u)$ *to be a positive definite M-matrix for each* x *and* u. *Then the solution* $x = x(u)$ *to problem* (2) *is monotone by* u.

The symmetrical result concerning the antitone behavior of solutions of the complementarity problem (2) is obtained readily by the theorem below.

Theorem 2. *Let* A *be a positive definite M-matrix,* B *a non-negative one, and* $\varphi(x,u)$ *a differentiable function monotone by both* x *and* u. *Moreover, suppose* $\varphi'_x = \varphi'_x(x,u)$ *to be a positive definite M-matrix for each* x *and* u. *Then the solution* $x = x(u)$ *to problem* (2) *is antitone by* u.

Extensions of the above-mentioned results to implicit complementarity problems can be found in [9]. The monotonicity of solutions to parametric variational inequalities, both in finite- and infinite-dimensional spaces, will be the object of the authors' future research.

References

1. Maier G. Problem 72-7*: A parametric linear complementarity problem // SIAM Review. 1972. V. 14, No. 2. P. 364–365. (1
2. Cottle R.W. Monotone solutions in parametric linear complementarity problems // Math. Programming. 1972. V. 3. No. 2. P. 210–224.
3. Megiddo N. On monotonicity in parametric linear complementarity problems// Math. Programming. 1977. V. 12. No. 1. P. 60–66.
4. Megiddo N. On the parametric nonlinear complementarity problem // Math. Programming Study. 1978. V. 17. No. 1. P. 142–150.
5. Kostreva M.M. Elasto-hydrodynamic lubrication: A nonlinear complementarity problem // Int. J. Numer. Methods Fluids. 1984. V. 4. No. 3. P. 377–397.
6. Ferris M.C., Pang J.-S. Engineering and economic applications of complementarity problems // SIAM Review. 1997. V. 39. No. 5. P. 669–713.
7. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York/London (1970).
8. Berman A., Plemmons D. Nonnegative Matrices in Mathematical Sciences. New York: Academic Press, 1979.
9. Kalashnikov V.V., Kalashnykova N.I., and Castillo-Pérez F.J. Solutions of parametric complementarity problems monotone with

respect to parameters // Journal of Global Optimization. 2016. V. 64. No. 4. P. 703–719.

Complexity estimates for one variant of the branch-and-bound algorithm for the subset sum problem*

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The subset sum problem is a particular case of the knapsack problem [1] stated as follows:

$$\begin{cases} f(x) = \sum_{i=1}^{n} w_i x_i \to \max, \\ \sum_{i=1}^{n} w_i x_i \le C, \\ x_i \in \{0,1\}, i \in \overline{1,n}. \end{cases} \tag{1}$$

Despite its simple formulation the problem is NP-hard. One of the most efficient methods for resolution of this problem is the Branch-and-Bound method with various elimination rules [1,2]. Though it is well known that advanced Branch-and-Bound methods can efficiently cope with hard subset sum instances the complexity theory was not enough elaborated.

We consider the Branch-and-Bound method with the dominance relation used to eliminate sub-problems. Let P is a sub-problem of the problem (1) obtained by fixing variables $x_1, \ldots, x_{\tau(P)}$, $\tau(P) \in \overline{0,n}$:

$$\begin{cases} f(x) = \sum_{i=1}^{n} w_i x_i \to \max, \\ \sum_{i=1}^{n} w_i x_i \le C, \\ x_i = \theta_i(P), i \in \overline{1, \tau(P)} \\ x_i \in \{0,1\}, i \in \overline{\tau(P)+1, n}. \end{cases} \tag{2}$$

Let us introduce the following designations

$$\begin{aligned} &C(P) = C - \textstyle\sum_{i=1}^{\tau(P)} \theta_i(P) w_i, \\ &\underline{k}(P) = \{\max k : k \in \overline{0, n-\tau(P)}, \textstyle\sum_{i=\tau(P)+1}^{\tau(P)+k} w_i \le C(P)\}, \\ &\overline{k}(P) = \{\max k : k \in \overline{0, n-\tau(P)}, \textstyle\sum_{i=n-k+1}^{n} w_i \le C(P)\}, \end{aligned}$$

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If $w_1 \geq w_2 \geq \cdots \geq w_n$ then obviously $\underline{k}(P) \leq \overline{k}(P)$. The sub-problem P is said to fulfill the *cardinality elimination rule* if $\underline{k}(P) = \overline{k}(P)$. In this case the optimal solution of P is readily available:

$$x_i^*(P) = \begin{cases} \theta_i(P), \text{ if } i \in \overline{1, \tau(P)}, \\ 1, \text{ if } i \in \overline{\tau(P)+1, \tau+\underline{k}(P)}, \\ 0, \text{ if } i \in \overline{\tau(P)+\underline{k}(P)+1, n}. \end{cases} \quad (3)$$

Therefore the sub-problem can be excluded from the further search after the incumbent solution is updated.

We say that the sub-problem P_1 is *equivalent* to the problem P_2 if $\tau(P_1) = \tau(P_2)$ and $C(P_1) = C(P_2)$. It is clear that equivalent sub-problems have the same objective value of the optimal solution and therefore only one equivalent sub-problems should be saved during the search process (the other one is eliminated). The introduced elimination rule is a particular case of the more general dominance relation for the knapsack problem [1].

After sorting the items in the non-increasing order, i.e. $w_1 \geq w_2 \geq \cdots \geq w_n$, the algorithm follows the standard branch-and-bound scheme. On each iteration it takes a sub-problem from the list, applies elimination rules and if the sub-problem is not eliminated it splits the sub-problem into smaller sub-problems by fixing the next free (non-fixed) variable. The *complexity* of the problem (1) is defined as the number of sub-problems considered by the algorithm described above during the resolution process.

The complexity bound for this problem is given by the following theorem.

Theorem. If $n \geq 3$ the worst case complexity for the problem (1) is $2\binom{n}{\lfloor \frac{n}{2} \rfloor} - 1$.

It is worth to note that the worst case complexity is reduced approximately twice by applying the cardinality elimination rule instead of the standard elimination rule while using of the equivalence elimination rule does not affect the upper bound.

References

1. Martello S., Toth P. Knapsack Problems. John Wiley & Sons Ltd., 1990.
2. Kellerer H., Pfershy U., Pisinger D. Knapsack Problems. Springer Verlag, 2004.

Selective bi-coordinate variations for optimization problems with simplex constraints*

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We consider a special class of optimization problems, where a goal function f is supposed to be smooth and a feasible set D is defined by simplex constraints. We write this problem as

$$\min_{x\in D} \to f(x), \tag{1}$$

where $D = \left\{x \in \mathbb{R}^n_+ \,\middle|\, \langle e, x\rangle = b\right\}$, b is a fixed (non-negative) number, e is the vector of units, $\mathbb{R}^n_+$ denotes the non-negative orthant in $\mathbb{R}^n$.

It is well known that many problems of optimal allocation of some resource within a system reduce to (1); see e.g. [1, 2]. In particular, they often arise in information and telecommunication networks; see e.g. [3]. Besides, similar optimization problems arise in machine learning, signal, speech and image recognition and processing, and related fields; see e.g. [4, 5]. These problems have huge dimensionality, their data may be very inexact and incomplete, but they do not require high accuracy of solutions. For this reason, we are interested in developing low cost iterative methods, which keep the convergence properties of the usual ones, but reduce the total computational expenses. Due to the simplex type constraints, the bi-coordinate iterative methods may appear rather efficient here. The first bi-coordinate method was proposed in [6]. The detailed description of its recent versions is given in [7].

In this work, we develop a selective bi-coordinate method with special threshold control and tolerances, which follows the approach suggested in [8]. It should be noted that this method can be treated as a self-adjustment process for attaining an equilibrium state of a general closed economic system; see [8, 9].

A point $\bar{x}$ is called a *stationary point* of (1) if

$$\bar{x} \in D,\ \forall i,j \in I = \{1,\ldots,n\}, i \neq j,\ \bar{x}_i > 0 \implies g_i(\bar{x}) \leq g_j(\bar{x}),$$

where $g_i(x) = \frac{\partial f(x)}{\partial x_i}$. Each solution of problem (1) is a stationary point, the reverse assertion is true if f is pseudoconvex.

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We now describe the iterative method for finding stationary points. Let $I_\varepsilon(x) = \{i \in I \mid x_i \geq \varepsilon\}$, $\mathbb{Z}_+$ denote the set of non-negative integers.

Method (BCV). *Initialization:* Choose a point $z^0 \in D$ and sequences $\{\delta_l\} \searrow 0$, $\{\varepsilon_l\} \searrow 0$. Set $l = 1$.
Basic cycle: Step 0: Set $k = 0$, $x^0 = z^{l-1}$.
Step 1: Choose an index $i \in I_{\varepsilon_l}(x^k)$ such that $g_i(x^k) - g_j(x^k) \geq \delta_l$ for some $j \in I$, set $d_i^k = -1$, $d_j^k = 1$, $d_s^k = 0$ for other indices $s \neq i, j$, and go to Step 2. Otherwise set $z^l = x^k$, $l = l + 1$ and go to Step 0. *(Restart)*
Step 2: Find m as the smallest number in $\mathbb{Z}_+$ such that

$$f(x^k + \theta^m x_i^k d^k) \leq f(x^k) + \beta\theta^m x_i^k \langle f'(x^k), d^k\rangle, \tag{2}$$

set $\lambda_k = \theta^m x_i^k$, $x^{k+1} = x^k + \lambda_k d^k$, $k = k + 1$ and go to Step 1.

The convergence properties of the method are formulated as follows.

Theorem 1. (a) For each stage l, the number of changes of index k in the basic cycle is finite;
(b) the sequence $\{z^l\}$ has limit points and all these points are stationary for (1);
(c) if f is pseudoconvex, then $\lim_{l\to\infty} f(z^l) = f^*$, and all the limit points of $\{z^l\}$ are solutions of (1).

The above descent method admits various modifications. Firstly, we can take the exact one-dimensional minimization rule instead of the current Armijo rule in (2). Secondly, if the function f is convex, we can replace (2) with the following:

$$\langle f'(x^k + \theta^m x_i^k c), d^k\rangle \leq \beta\theta^m x_i^k \langle f'(x^k), d^k\rangle,$$

where only two selected coordinates of d^k are nonzero. Next, if the gradient of the function f possesses even partial Lipschitz continuity properties, we can simply take the fixed stepsize.

Moreover, given a starting point z^0 and a number $\alpha > 0$, we can evaluate the complexity of the method in this case. It is defined as the total number of iterations at $l(\alpha)$ stages such that $l(\alpha)$ is the maximal number l with $f(z^l) - f^* \geq \alpha$ and denoted by $N(\alpha)$, where $f^* = \inf\limits_{x\in D} f(x)$. If the function f is convex with Lipschitz continuous partial gradients, then the method attains the complexity estimate $N(\alpha) = O(1/\alpha)$; see [10].

In computational tests, (BCV) showed rather rapid convergence in comparison with the known methods such as the conditional gradient method and bi-coordinate descent methods with random and marginal estimate rules for selection of coordinate indices. In particular, it reduces the total volume of computational expenses in comparison with the conditional gradient method since it does not require calculations of all the partial derivatives at each iteration in general. At the same time, (BCV) is suitable for parallel and distributed (multi-agent) computations.

The method admits extensions to the more general classes of problems, which involve both lower and upper bounds for variables, besides, the equality constraint $\langle e, x\rangle = b$ can be replaced by $\langle a, x\rangle = b$, where a is an arbitrary vector in $\mathbb{R}^n$ and b is an arbitrary number.

References

1. Konnov I.V. Equilibrium Models and Variational Inequalities. Amsterdam: Elsevier, 2007.
2. Patriksson M. A survey on the continuous nonlinear resource allocation problem // Eur. J. Oper. Res. 2008. V. 185, № 1. P. 1–46.
3. Stańczak S., Wiczanowski M., Boche H. Resource Allocation in Wireless Networks. Theory and Algorithms. Berlin: Springer, 2006.
4. Burges C.J.C. A tutorial on support vector machines for pattern recognition// Data Mining Know. Disc. 1998. V. 2, №2. P.121–167.
5. Cevher V., Becker S., Schmidt M. Convex optimization for big data // Signal Process. Magaz. 2014. V.31, №5. P.32–43.
6. Korpelevich G.M. Coordinate descent method for minimization problems with linear inequality constraints and matrix games// Mathematical Methods for Solving Economic Problems. V. 9. Moscow: Nauka, 1980. P. 84–97.
7. Beck A. The 2-coordinate descent method for solving double-sided simplex constrained minimization problems // J. Optim. Theory. Appl. 2014. V. 162, №3. P.892–919.
8. Konnov I.V. Selective bi-coordinate variations for resource allocation type problems// Comp. Optim. Appl. DOI 10.1007/s10589-016-9824-2
9. Konnov I.V. An alternative economic equilibrium model with different implementation mechanisms// Adv. Model. Optim. 2015. V.17, №2. P.245–265.

10. Konnov I.V. A method of bi-coordinate variations with tolerances and its convergence// Russ. Mathem. (Iz. VUZ). 2016. V. 60, №1. P.68–72.

Normative dynamic analysis of a heterogeneous computing system

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Consider the heterogeneous computer systems (CS) that processes a flow of various computationally intensive tasks under uncertainty (CITUs). To improve performance, a CS has specialized units that considerably speed-up of some procedures by compare with a general-purpose processor. Different types of units completed the same task in a different amount of time; moreover, some can execute only specific types of algorithms and are applicable to a limited class of tasks. To use the CS resources more efficiently and satisfy the principle of equal significance of tasks when the CITUs are scheduled, optimization models and approaches are used (see [1-3]) that form a hardware and software environment. In this report, we study the operation of a heterogeneous computer system from the viewpoint of its performance. Typically, performance is defined as the amount of computational work performed in a unit of time or during a time interval.

The number and performance of processing units in a heterogeneous CS may change with time; moreover, new versions of software and control subsystems can drastically change the amount and the whole set of works. Hence, it is the problem of analysis of the CS functional capabilities dynamics under the conditions of changes in the workability of elements due to failures.

In this report make use a multiparameter model (MP model) to analyze the dynamics of a CS performance based on deriving guaranteed bounds on the amount of work that can be accomplished provided that the resources are allocated efficiently. The input task flow is intensive, and the CS can complete only a part of these tasks. As the characteristic of the CS functional capabilities make use of the vector of simultaneously executed tasks. The components of this vector correspond to the amount of computational work that can be jointly completed in one operational window. Each feasible allocation of available resources is assigned a vector consisting of the set of executed tasks, and the points at the boundary of this set determine the extreme functional capabilities of the CS. To investigate these boundaries and vectors, single-task operational modes are considered in which the system processes only a single type of tasks.

The maximum functional capabilities of the CS are determined by solving the following multiple criteria optimization problem: maximize the vector of executed tasks on the set of feasible resource allocations. The values of the maximum amount of work that can be done in the single-task mode of task processing are used as weighting coefficients in the multiple criteria optimization of resource allocation. The maximum functional capabilities are described by a subset of Pareto optimal vectors of executed tasks (none of the components of such a vector can be increased without decreasing another component).

For a fully operational CS working at its maximum performance, the concept of the initial normal state is introduced. The normative functional characteristics are determined by the Pareto optimal solution to the problem for the initial normal state of the system, which is determined by the weighting coefficients obtained for the single-task operational modes of the system.

To find a dynamic estimate of the CS state in the beginning of each operational window taking into account the actual state of resources, the current limiting functional characteristics are computed.

At certain check time point, the current maximum values of the performance indicators are compared with the normative ones. A two-dimensional diagram of relative deviations is constructed, which makes it possible to track the dynamics of performance indicator changes.

The constructed charts illustrate variations in the limiting functional characteristics when the technical characteristics of the system elements vary. The analysis of the charts obtained over a long time period makes it possible to reliably estimate the functional capabilities of the system in various operational states (hardware failures) when the system processes tasks of different types.

References

1. Yu.E. Malashenko and I. A. Nazarova. Control of resource intensive computations under uncertainty. I. Multiparametric model // J. Comput. Syst. Sci. Int. 2014. V. 53, № 4. P. 497–510.
2. I. K. Kupalov-Yaropolk, Yu. E. Malashenko, I. A. Nazarova, and A. F. Ronzhin. Control of resource intensive computations under uncertainty. II. Scheduling complex // J. Comput. Syst. Sci. Int.

2014. V. 53, № 5. P. 636–644.

3. Yu. E. Malashenko and I. A. Nazarova. Control of resource intensive computations under uncertainty. III. Dynamic concurrent resource allocation // Comput. Syst. Sci. Int. 2015. V. 54, № 1. P. 48–58.

On methods for solving quasi variational inequalities

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1. Introduction. Consider the following quasi variational inequality: find $x_* \in C(x_*)$ such that

$$\langle F(x_*), y - x_* \rangle \geq 0 \ \ \forall y \in C(x_*), \tag{1}$$

where $C : H \to 2^H$ is set-valued mapping with nonempty convex and closed set $C(x) \subseteq H$ for all x from Hilbert space H.

Note that the difficulty of problems with quasi variational inequalities is related to the fact that one must simultaneously solve a variational inequality and calculate a fixed point of a set-valued mapping. This explains why the literature on solution methods for quasi variational inequalities is not too extensive. Consequently, there are numerous open questions.

We will suppose that the operator F satisfies the Lipschitz condition with the positive constant L and strong monotonicity with positive constant μ.

The theorems about existence of solutions show a notable difference between variational and quasi variational inequalities. For example, if F is strongly monotone and Lipschitz continuous on closed and convex set, then variational inequality has a unique solution. On the other hand, for quasi variational inequalities it is necessary to add a condition (see [1,2]):

$$\|\Pi_{C(x)}[z] - \Pi_{C(y)}[z]\| \leq \lambda \|x - y\|, \ \ \forall x, y, z \in H, \ \ \lambda < \frac{\mu}{L}, \tag{2}$$

where $\Pi_C[z]$ is the projection of point z onto the set C.

In many important applications the convex valued set $C(x)$ can be written as $C(x) = c(x) + C_0$, where C_0 is a closed convex set and $c : H \to H$ is a Lipschitz continuous mapping with constant $\lambda > 0$. In this case, assumption (2) holds with the same value of λ (see [2]).

Example 1. Mapping $F(x) = x, x \in R$ is strongly monotone and Lipshitz continuous with constants $\mu = L = 1$. Then, for $C(x) = \{x\}$, or $C(x) = [x, x+1]$ quasi variational inequality: find $x_* \in C(x_*)$ such that $\langle F(x_*), y - x_* \rangle \geq 0 \ \forall y \in C(x_*)$, has infinitely many solutions (the set of solutions is R). If $C(x) = [x-1, x]$ the set of solutions of this inequality is empty.

Example 2. If

$$C(x) = \begin{cases} [1/2, 1], & \text{if } \ x \in [0, 1/2) \\ [0, 1], & \text{if } \ x = 1/2 \\ [0, 1/2], & \text{if } \ x \in (1/2, 1] \end{cases}$$

mapping C has a unique fixed point $x_* = 1/2$, but it is not a solution of (1).

2. Continuous methods. We will consider the differential equation

$$x'(t) + x(t) = \Pi_{C(x(t))}[x(t) - \alpha(t)F(x(t)], \ \ t \geq 0, x(0) = x_0, \qquad (3)$$

where x_0 is a given initial point in H and $\alpha > 0$ is a parameter of the method. Then, solution x_* of quasi variational inequality (1) is a stationary point of system (3).

Theorem 1. *Let operator $F : H \to H$ be strongly monotone (with constant $\mu > 0$) and Lipshitz continuous (with constant $L > 0$), set-valued mapping $C : H \to 2^H$ with nonempty, closed and convex values $C(x) \subseteq H \ \forall x \in H$ satisfies condition (2) and parameter $\alpha(t) \in C([0, +\infty)$ satisfies the following conditions: $0 < \alpha_0 \leq \alpha(t) \leq \alpha_1, \ \ \forall t \geq 0$, where $\alpha_0 > \frac{\mu - \sqrt{\mu^2 - L^2(2\lambda - \lambda^2)}}{L^2}$, $\alpha_1 < \frac{\mu + \sqrt{\mu^2 - L^2(2\lambda - \lambda^2)}}{L^2}$. Then, for all $x_0 \in H$, the trajectory $x(t)$, $t \geq 0$ defined by (3) converges to the unique solution $x_* \in C(x_*)$ of problem (1) with the following rate:*

$$\|x(t) - x_*\| \leq e^{-a_0 t/2} \|x_0 - x_*\|,$$

$$\text{where } a_0 = 1 - \left(\lambda + \sqrt{1 - 2\alpha_1 \mu + \alpha_0^2 L^2}\right)^2.$$

Continuous proximal method for quasi variational inequalities was considered in [5].

3. Iterative methods. Some iterative versions of the gradient projection method for convex minimization, variational and quasi variational inequalities were investigated in [2,3]. Here, we describe iterative proximal method, which can be understood as an implicit variant of the gradient projection method. Let $x_0 \in C_0$ be an arbitrary initial approximation of the solution. Suppose that, for a certain $k > 0$, the approximation $x_k \in C(x_{k-1})$ has already been determined. Then the set $C(x_k)$ is defined. The approximation $x_{k+1} \in C(x_k)$ is determined as a solution to the following variational inequality: find x_{k+1} for which

$$\langle x_{k+1} - x_k + \alpha F(x_{k+1}), z - x_{k+1} + c(x_k)\rangle \geq 0, \quad \forall z \in C_0. \tag{4}$$

where $\alpha > 0$. Note that this inequality is uniquely solvable. Method is described. In the theorem below, we state conditions for the convergence of this method and estimate the convergence rate.

Theorem 2. *Let the following assumptions be fulfilled:*

(1) $C_0 \subseteq H$ is a convex closed subset of the Hilbert space H, $c : H \to H$ is a Lipschitz continuous operator with the constant $l > 0$ and $C : H \to 2^H$ is a set-valued mapping of the form $C(x) = c(x) + C_0$, $x \in H$;

(2) The operator $F : H \to H$ is strongly monotone with the constant $\mu > 0$ and Lipshitz continuous with the constant $L > 0$;

(3) The parameter α and the constants l, L, and μ satisfy the conditions $l < \frac{\sqrt{2}}{2}\frac{\mu}{L}$, $\left|\alpha - \frac{\mu}{L^2}\right| < \frac{1}{L^2}\sqrt{\mu^2 - 2l^2L^2}$. Then, for every initial approximation $x_0 \in C_0$, the sequence $\{x_k\}$ defined by (4) converges to the unique solution $x_* \in C(x_*)$ of problem (1). Moreover, the following estimate for the convergence rate is valid:

$$\|x_{k+1} - x_*\| \leq q^k(\alpha)\|x_0 - x_*\|, \text{ where } q(\alpha) = \sqrt{\frac{1 + 2l^2}{1 + 2\alpha\mu - \alpha^2L^2}}.$$

References

1. Noor M. A., Oettli W., On general nonlinear complementarity problems and quasi equilibria. Le Mathematiche XLIX, 1994., p. 313-331,
2. Nesterov Yu., Scrimali L., Solving strongly monotone variational and quasi-variational inequalities, Core discussion paper, 2006/107,
3. Vasiliev F. P., Methods of Optimization, Moscow, MCCME, (2011) (in Russian)
4. Jaćimović M., Mijajlović N., On a Continuous Gradient-type Method for Solving Quasi-variational Inequalities. Proceedings of the Montenegrin Academy of Sciences and Arts, Vol. 19, (2010)

5. Mijajlović, N., Jaćimović, M.: Proximal methods for solving quasi-variational inequalities, Computational Mathematics and Mathematical Physics, Vol. 55, No. 12, pp. 1981–1985, (2015)

Optimization methods and software for seeking a Nash equilibrium in hexamatrix games*

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Consider the following polymatrix game of three players (hexamatrix game) with mixed strategies:

$$\left.\begin{array}{r} F_1(x,y,z) \triangleq \langle x, A_1 y + A_2 z\rangle \uparrow \max\limits_{x},\ x \in S_m, \\ F_2(x,y,z) \triangleq \langle y, B_1 x + B_2 z\rangle \uparrow \max\limits_{y},\ y \in S_n, \\ F_3(x,y,z) \triangleq \langle z, C_1 x + C_2 y\rangle \uparrow \max\limits_{z},\ z \in S_l, \end{array}\right\}$$

where $S_p = \{(u_1,\ldots,u_p)^T \in I\!R^p \mid u_i \geq 0,\ \sum\limits_{i=1}^{p} u_i = 1\}, \quad p = m,n,l.$

Further consider the following nonconvex optimization problem ($\sigma \triangleq (x,y,z,\alpha,\beta,\gamma)$):

$$\left.\begin{array}{c} \Phi(\sigma) \triangleq \langle x, A_1 y + A_2 z\rangle + \langle y, B_1 x + B_2 z\rangle + \langle z, C_1 x + C_2 y\rangle - \\ -\alpha - \beta - \gamma \uparrow \max\limits_{\sigma},\ \sigma \in D \triangleq \{(x,y,z,\alpha,\beta,\gamma) \in I\!R^{m+n+l+3} \mid \\ \mid x \in S_m,\ \ y \in S_n,\ \ z \in S_l,\quad A_1 y + A_2 z \leq \alpha e_m, \\ B_1 x + B_2 z \leq \beta e_n,\quad C_1 x + C_2 y \leq \gamma e_l\}, \end{array}\right\} \quad (\mathcal{P})$$

where $e_p = (1,1,...,1) \in I\!R^p, p = m,n,l.$

The search for a global solution to Problem $(\mathcal{P})$ is equivalent to a finding Nash equilibria in hexamatrix game [1] constructed with matrices $A = (A_1, A_2)$, $B = (B_1, B_2)$, and $C = (C_1, C_2)$.

Theorem. [1] *A point (x^*, y^*, z^*) is a Nash equilibrium point in the hexamatrix game $\Gamma(A,B,C)$ if and only if it is a part of a global solution $\sigma_* \triangleq (x^*, y^*, z^*, \alpha_*, \beta_*, \gamma_*) \in I\!R^{m+n+l+3}$ of Problem $(\mathcal{P})$. At the same time, the numbers α_*, β_*, and γ_* are the payoffs of the first, the second, and the third players, respectively, in the game $\Gamma(A,B,C)$:*

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$\alpha_* = v_1(x^*, y^*, z^*)$, $\beta_* = v_2(x^*, y^*, z^*)$, $\gamma_* = v_3(x^*, y^*, z^*)$. *In addition, an optimal value* $\mathcal{V}(\mathcal{P})$ *of Problem* $(\mathcal{P})$ *is equal to zero:* $\mathcal{V}(\mathcal{P}) = \Phi(\sigma_*) = 0$.

In order to solve Problem $(\mathcal{P})$, we are using an approach based on Global Search Theory [2]. According to this theory the Global Search consists of two principal stages: 1) a local search, which takes into account the structure of the problem under scrutiny; 2) the procedures based on Global Optimality Conditions (GOC) [2], which allow to improve the point provided by the local search method, in other words, to escape a local pit.

To implement a local search in Problem $(\mathcal{P})$, we are applying the ideas, first, of splitting variables in several groups, and, after that, of consecutive solving of specially constructed LP problems with respect to the groups of variables. These ideas have previously demonstrated its efficiency in bimatrix games [3], bilinear programming problems [3], and bilevel problems [4].

In order to do it, consider the following LP problems:

$$\left.\begin{array}{c} f_1(x,\beta) \triangleq \langle x, (A_1 + B_1^T)v + (A_2 + C_1^T)w\rangle - \beta \uparrow \max\limits_{(x,\beta)}, \\ (x,\beta) \in X(v,w,\bar{\gamma}) \triangleq \{(x,\beta) \mid x \in S_m, \\ B_1 x - \beta e_n \leq -B_2 w,\ C_1 x \leq \bar{\gamma} e_l - C_2 v\}; \end{array}\right\} \qquad (\mathcal{LP}_x(v,w,\bar{\gamma}))$$

$$\left.\begin{array}{c} f_2(y,\gamma) \triangleq \langle y, (B_1 + A_1^T)u + (B_2 + C_2^T)w\rangle - \gamma \uparrow \max\limits_{(y,\gamma)}, \\ (y,\gamma) \in Y(u,w,\bar{\alpha}) \triangleq \{(y,\gamma) \mid y \in S_n, \\ A_1 y \leq \bar{\alpha} e_m - A_2 w,\ C_2 y - \gamma e_l \leq -C_1 u\}; \end{array}\right\} \qquad (\mathcal{LP}_y(u,w,\bar{\alpha}))$$

$$\left.\begin{array}{c} f_3(z,\alpha) \triangleq \langle z, (C_1 + A_2^T)u + (C_2 + B_2^T)v\rangle - \alpha \uparrow \max\limits_{(z,\alpha)}, \\ (z,\alpha) \in Z(u,v,\bar{\beta}) \triangleq \{(z,\alpha) \mid z \in S_l, \\ A_2 z - \alpha e_m \leq -A_1 v,\ B_2 z \leq \bar{\beta} e_n - B_1 u\}. \end{array}\right\} \qquad (\mathcal{LP}_z(u,v,\bar{\beta}))$$

Here $(u, v, w, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in D$ is a feasible point in Problem $(\mathcal{P})$.

The local search method based on a consecutive solving of these LPs converges to the point $\hat{\sigma} \triangleq (\hat{x}, \hat{y}, \hat{z}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$, which is satisfying the following inequalities:

$$\Phi(\hat{\sigma}) \geq \Phi(x, \hat{y}, \hat{z}, \hat{\alpha}, \beta, \hat{\gamma}) \quad \forall (x,\beta) \in X(\hat{y}, \hat{z}, \hat{\gamma}),$$

$$\Phi(\hat{\sigma}) \geq \Phi(\hat{x}, y, \hat{z}, \hat{\alpha}, \hat{\beta}, \gamma) \quad \forall (y,\gamma) \in Y(\hat{x}, \hat{z}, \hat{\alpha}),$$

$$\Phi(\hat{\sigma}) \geq \Phi(\hat{x}, \hat{y}, z, \alpha, \hat{\beta}, \hat{\gamma}) \quad \forall (z, \alpha) \in Z(\hat{x}, \hat{y}, \hat{\beta}).$$

Such a point point can be called a partially global solution of the problem $(\mathcal{P})$ (with respect to pairs (x, β), (y, γ), and (z, α)).

For a global search procedure, first, we need to construct the explicit representation of the objective function Φ as a difference of two convex functions, for example, as follows:

$$\Phi(x, y, z, \alpha, \beta, \gamma) = h(x, y, z) - g(x, y, z, \alpha, \beta, \gamma),$$

$$h(x, y, z) = \frac{1}{4}\Big(\|x + A_1 y\|^2 + \|x + A_2 z\|^2 + \|B_1 x + y\|^2 + \|y + B_2 z\|^2 + \|C_1 x + z\|^2 + \|C_2 y + z\|^2\Big), \quad g(\sigma) = \frac{1}{4}\Big(\|x - A_1 y\|^2 + \|x - A_2 z\|^2 + \|B_1 x - y\|^2 + \|y - B_2 z\|^2 + \|C_1 x - z\|^2 + \|C_2 y - z\|^2\Big) + \alpha + \beta + \gamma.$$

Therefore, the global search method in Problem $(\mathcal{P})$ is based on GOC for d.c. maximization problems (see [2-4]). According to [2-4], the global search procedure consists of several stages such as constructing an approximation of the level surface of the convex function $h(x, y, z)$, which generates a basic nonconvexity of the problem $(\mathcal{P})$, solving the linearized convex problem, an implementing of additional local search, verifying GOC etc. As a result, taking into account the features of Problem $(\mathcal{P})$ and using all the stages of the global search above mentioned, we have constructed and implemented the Global Search Algorithm in the hexamatrix games.

The software, implementing elaborated methods of local and global search has been developed in MATLAB 7.11.0.584 R2010b. As for auxiliary LP problems and convex quadratic problems, they have been solved by corresponding MATLAB subroutines of famous software package IBM CPLEX (v. 12.62). This package shows the considerable advantages with respect to standard MATLAB subroutines "linprog" and "quadprog".

The efficiency of created software is demonstrated by the results of computational solving of the large amount of test hexamatrix games.

References

1. Strekalovsky A.S., Enkhbat R. Polymatrix games and optimization problems // Automation and Remote Control. 2014. V. 75, No. 4, P. 632–645.
2. Strekalovsky A.S. Elements of nonconvex optimization. Novosibirsk: Nauka, 2003 (in Russian).

3. Strekalovsky A.S., Orlov A.V. Bimatrix games and bilinear programming. Moscow: FizMatLit, 2007 (in Russian).
4. Strekalovsky A.S., Orlov A.V., Malyshev A.V. On computational search for optimistic solutions in bilevel problems // Journal of Global Optimization. 2010. V. 48, No. 1, P. 159–172.

Stability of a model predictive impulsive control scheme*

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This article concerns the stability of an optimal control based receding horizon scheme - often referred to by Model Predictive Control (MPC) - for dynamic impulsive control systems. An optimizing framework for state feedback control of the dynamic system emerges from the articulation of a discrete-time state sampling strategy with the control synthesis via optimality conditions, notably, necessary conditions of optimality in the form of a Maximum Principle (see [1,2], and, then, appropriately sliding the time horizon. Unlike [3], this is a practical approach that combines optimality conditions with state-variable sampling in order to take into account perturbations that affect the behavior of real-world systems, while mitigating the huge computational burden typically associated with the on-line computation of optimal feedback control, which, in general, requires solving a certain Hamilton-Jacobi-Bellman partial differential equation, [4]. There is not only an abundant body of literature on MPC schemes for conventional control - systems with absolutely continuous trajectories and references therein, but also, it has been widely used by the control practitioners for a significant period of time now, [5].

This state-of-affairs strongly contrasts with the one for impulsive control systems, that is, dynamic systems whose control space is enlarged to contain measures and, thus, the associated trajectories are merely of bounded variation, and, in particular, may have jumps. In particular, we consider systems of the form

$$dx = f(t, x, u)dt + G(t, x, u)d\vartheta$$

with $(x(0), x(T)) \in C_0 \times C_T$, $u \in \mathcal{U}$, and $\vartheta \in \mathcal{I}$, where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times k}$ are given mappings, C_0 and

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C_T are compact sets, $\mathcal{U} = \{u \in L_\infty([0,T];\mathbb{R}^m) : u(t) \in \Omega\}$, with compact $\Omega \subset \mathbb{R}^m$, $\mathcal{I}$ is the impulsive control constraint set, and $\vartheta = (\mu, \{u_\tau, v_\tau\})$ is the impulsive control which is specified by two components: a Borel measure $\mu \in \mathcal{K}$ with range in convex, closed and pointed cone K in $\mathbb{R}^k$, and a certain pair of functions $\{u_\tau, v_\tau\}$ defined on the support of the atomic component of μ. For details, see [6,7].

Intuitively, the need to adopt an impulsive control framework arises when the control systems exhibits very fast and very slow dynamics abstraction and the optimal control problem of interest is such that these two components of the dynamics can not be dealt with separately. There are several concepts of impulsive control and impulsive trajectories in the literature. We consider the ones defined in [6] which, arguably, are among the most sophisticated ones in that it is well suited to capture the requirements of important classes of engineering systems, [7].

The MPC scheme for the this class of impulsive control systems proposed here is a refinement of the one described in Chapter 9 (An Optimization-based Framework for Impulsive Control Systems) in [5] and it enables the construction of a state feedback control law by jointly computing sequences of

- sampling instants $\pi := \{t_i\}_{i \geq 0}$ in $[0, +\infty)$ with inter-sampling times $\delta_i > 0$ such that $t_{i+1} = t_i + \delta_i$ for all $i \geq 0$,
- open loop optimal controls on $[t_i, t_i + T]$ by solving the optimal control problems $\mathcal{P}(t_i, x_i, T)$ at each sampling instant $t_i \in \pi$ by using the current measure of the state variable $x(t_i) = x_i$,

where

$$\begin{aligned}
\mathcal{P}(t_i, x_i, T) \text{ Minimize} \quad & W(t_i + T, x(t_i + T)) + \int_{t_i}^{t_i+T} L_{ac}(s, x(s), u(s))ds \\
& \qquad + \int_{[t_i, t_i+T]} L_s(s, x(s), u(s))d\vartheta(s) \\
\text{subject to} \quad & dx(t) = f(t, x(t), u(t))dt + G(t, x(t), u(t))d\vartheta(t) \\
& \qquad \forall t \in [t_i, t_i + T], \\
& u \in \mathcal{U}_{|[t_i, t_i+T]}, \ \vartheta \in \mathcal{I}_{|[t_i, t_i+T]}, \ x(t_i + T) \in S,
\end{aligned}$$

The proposed MPC scheme involves a form of receding horizon that takes into account the specificities of the impulsive control, is a follows:

1. Initialization. Set parameters, specify initial data, and iteration counter $i = 0$.
2. Sample the current state of the plant $x(t_i) = x_i$.

3. Solve problem $\mathcal{P}(t_i, x_i, T)$ to obtain the open-loop optimal conventional control $\bar{u}^i \in \mathcal{U}_{|[t_i,t_i+T]}$ and impulsive control $\bar{\vartheta}^i \in \mathcal{I}_{|[t_i,t_i+T]}$. Whenever $\bar{\mu}^i(\{t\}) \neq 0$ (i.e., if the optimal control measure has an atom, including the time endpoints t_i and $t_i + T$), then, the optimal arc joining the associated trajectory endpoints has to be defined by computing the optimal pair of functions $(\bar{u}^i_t(\cdot), \bar{v}^i_t(\cdot))$ defined on the associated emerging interval $[t, t + |\bar{\mu}^i(\{t\})|]$.
4. Determination of the next sampling instant. This is the earliest time in which either a time interval of duration δ elapses, or an atom of $\bar{\mu}^i$ occurs. We remark that the case of $\tau = 0$ makes sense when the perturbations affecting the system are extremely fast and an abstract "set-valued sampling rate" is considered.
5. Apply to the plant the control pair $\bar{u}^i$ and $\bar{\vartheta}^i$ during the interval $[t_i, t_i+\delta_i]$, being the control strategy values computed for $t \geq t_i+\delta_i$ discarded.
6. Now the optimization time horizon slides, i.e., we consider $t_{i+1} = t_i + \delta_i$, we let $i = i + 1$ and repeat the procedure from step 2.

Here, the closed set $S \subset \mathbb{R}^n$, and the mappings $W : [t_i, t_i+T] \times \mathbb{R}^n \to \mathbb{R}$, $L_{ac} : [t_i, t_i+T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $L_s : [t_i, t_i+T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ are chosen in order to ensure the stability of the MPC scheme. Under mild assumption on the data of the impulsive control system, a Lyapunov like asymptotic stability of this MPC scheme are proved in the context of nonsmooth context, [2,8], and by making use of auxiliary extension of invariance results, [2], for impulsive systems.

References

1. Arutyunov A. V. Optimality Conditions: Abnormal and Degenerate Problems. Kluwer Academic Publishers, 2000.
2. Vinter R. B. Optimal Control. Birkhauser Boston, 2000.
3. Pereira F., Silva G. N. Lyapunov stability of measure driven impulsive systems // Differential Equations. 2004. V. 40. P. 1122–1130
4. Fraga S. L., Pereira F. Hamilton-Jacobi-Bellman Equation and Feedback Synthesis for Impulsive Control // IEEE Trans. on Autom. Control. 2012. V. 57. P. 244–249
5. Olaru S., A. Grancharova A., Pereira F. (ed.). Developments in Model-Based Optimization and Control Distributed Control and Industrial Applications. Lect Notes in Control and Inf. Sci. 464, Springer, 2016.

6. Arutyunov A. V., Karamzin D. Y., Pereira F. Pontryagin's maximum principle for constrained impulsive control problem // Nonlin. Anal.-Theory, Method & Appl. 2012. V. 75. P. 1045—1057
7. Arutyunov A. V., Karamzin D. Y., Pereira F. Impulsive Control Problems with State Constraints: R.V. Gamkrelidze Approach to the Necessary Optimality Conditions // J. of Optim Theory & Appl. 2014. V. 166. N. 2. P. 440–459
8. Mordukhovich B. S. Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications. Springer, Berlin, 2006.

On smooth approximation of convex sets and convex functions*

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1. Smooth approximations of convex sets.

Let a set $X \subset \mathbb{R}^n$ be closed and convex and $x \in X$. A closed convex set is called *smooth* if at each of its boundary point there exists a unique support hyperplane.

A set

$$N(X,x) = \left\{ g \in \mathbb{R}^n \;\middle|\; \langle g, z - x\rangle \leqslant 0 \quad \forall z \in X \right\}$$

is called *the normal cone* to X at a point $x \in X$ [1]. $N(X,x)$ is closed and convex.

Thus if the normal cone at each boundary point $x \in X$ consists of a single ray then the set X is smooth.

Let $X \subset \mathbb{R}^n$ be closed and convex set and do not coincide with $\mathbb{R}^n$. Consider the closed convex set

$$Z_\varepsilon = X + \varepsilon B_1(0_n), \qquad \varepsilon > 0,$$

where

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid ||x - x_0|| \leqslant r\}.$$

Hereinafter, $||x|| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm. Note that Z_ε is the set with nonempty interior under every positive ε.

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Theorem 1. *The normal cone to Z_ε at every boundary point $z_0 \in$ bd (Z_ε) consists of a single ray.*

2. Smooth approximations of convex functions.

Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions.

A function

$$f(x) = \inf_{\substack{x_1 + x_2 = x \\ x_1, x_2 \in \mathbb{R}^n}} \{f_1(x_1) + f_2(x_2)\} = \inf_{x_1 \in \mathbb{R}^n} \{f_1(x_1) + f_2(x - x_1)\}$$

is called *infimal convolution* of two functions f_1 and f_2 and is denoted by

$$f(x) = (f_1 \oplus f_2)(x).$$

The function f is convex. The operation of taking the infimal convolution of two convex functions is commutative and associative.

Fix $\varepsilon > 0$. Define a function

$$t_\varepsilon(x) = \begin{cases} -\sqrt{\varepsilon^2 - \langle x, x\rangle}, & ||x|| \leqslant \varepsilon, \\ +\infty, & ||x|| > \varepsilon, \end{cases} \qquad x \in \mathbb{R}^n.$$

Note that

$$t^*_\varepsilon(v) = \varepsilon\sqrt{1 + \langle v, v\rangle}, \quad v \in \mathbb{R}^n,$$

where t^*_ε is the conjugate function of t_ε.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $D \subset \mathbb{R}^n$ be closed convex set. Denote

$$X = \left\{[x, \mu] \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \mu \geq f(x), \quad x \in D\right\}.$$

Construct families of smooth closed convex sets $\{Z_\varepsilon\}$, $\{D_\varepsilon\}$,

$$Z_\varepsilon = X + \varepsilon B_1(0_{n+1}) \subset \mathbb{R}^{n+1}, \quad \varepsilon > 0,$$

$$D_\varepsilon = D + \varepsilon B_1(0_n) \subset \mathbb{R}^n,$$

and a family of convex functions $\{f_\varepsilon\}$,

$$f_\varepsilon(x) = \begin{cases} \inf \mu, & [x, \mu] \in Z_\varepsilon \\ +\infty, & \text{в остальных случаях.} \end{cases}$$

It is not difficult to note that dom $f_\varepsilon = D_\varepsilon$ and for every fixed $\varepsilon > 0$ the graph of $f\varepsilon$ is a lower envelope of the set X_ε.

Fix $\varepsilon > 0$. Let $z \in D$. Consider a family of convex functions $\{\varphi_\varepsilon(x,z)\}$

$$\varphi_\varepsilon(x,z) = f(z) + t_\varepsilon(x,z),$$

where

$$t_\varepsilon(x,z) = \begin{cases} -\sqrt{\varepsilon^2 - ||x-z||^2}, & x \in a_\varepsilon(z), \\ +\infty, & \text{in other cases.} \end{cases}$$

Here

$$a_\varepsilon(z) = \{x \in \mathbb{R}^n \mid ||x-z|| \leqslant \varepsilon\} \subset D_\varepsilon.$$

It's obvious that

$$\mathrm{dom}\ \varphi_\varepsilon(\cdot, z) = a_\varepsilon(z), \quad \bigcup_{z \in D} a_\varepsilon(z) = D_\varepsilon.$$

Denote $H_\varepsilon(z) = \mathrm{epi}\ \varphi_\varepsilon(\cdot, z)$. Consider also functions

$$\varphi_\varepsilon(x) = \inf_{z \in D} \varphi_\varepsilon(x,z)$$

and their epigraphs $H_\varepsilon = \mathrm{epi}\ \varphi_\varepsilon$.

Theorem 2. *The following relations*

1. $f_\varepsilon(x) = (f \oplus t_\varepsilon)(x) = \varphi_\varepsilon(x)$,

2. *dom* $f_\varepsilon =$ *dom* $f_1 + B_\varepsilon(0_n)$, *epi* $f_\varepsilon =$ *epi* $f_1 + B_\varepsilon(0_{n+1})$,

where

$$B_\varepsilon(0_n) = \{x \in \mathbb{R}^n \mid ||x|| \leqslant \varepsilon\}, \quad B_\varepsilon(0_{n+1}) = \{x \in \mathbb{R}^{n+1} \mid ||x|| \leqslant \varepsilon\},$$

hold.

Theorem 3. *The function f_ε is continuously differentiable at every interior point of the set D_ε for each fixed $\varepsilon > 0$.*

Theorem 4. *The set epi f_ε is smooth for each fixed $\varepsilon > 0$.*

Denote by M a set of minimizers of the function f on the set D, and denote by M_ε a set of minimizers of the function f_ε on the set D_ε. The case in which these sets are empty is not excluded.

Theorem 5.

1. *The equality $M = M_\varepsilon$ holds.*

2. *If M is a nonempty then*

$$f_\varepsilon(z^*) = f(z^*) - \varepsilon \quad \forall z^* \in M.$$

References

1. Rockafellar R.T. Convex Analysis. Princeton. New York: Princeton Univ. Press. 1970.
2. Leichtweiss K. Konvexe Mengen. Berlin. 1980.

A two-step proximal algorithm of solving the problem of equilibrium programming

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Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction with $F(x,x) = 0$ for all $x \in C$. Consider the following equilibrium problem in the sense of Blum and Oettli [1, 2]:

$$\text{find} \quad x \in C \quad \text{such that} \quad F(x,y) \geq 0 \quad \forall\, y \in C.$$

We propose a new iterative two-step proximal algorithm for solving the problem of equilibrium programming in a Hilbert space. This method is a result of extension of L. D. Popov's modification of Arrow-Hurwicz scheme for approximation of saddle points of convex-concave functions [3, 4]. More precisely, we propose and analyse the following algorithm: for $x_1,\ y_1 \in C$ generate the sequences $x_n,\ y_n \in C$ with the iterative scheme

$$\begin{cases} x_{n+1} \in \operatorname{prox}_{\lambda F(y_n,\cdot)} x_n = \operatorname{argmin}_{y \in C} \left\{ \lambda F(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}, \\ y_{n+1} \in \operatorname{prox}_{\lambda F(y_n,\cdot)} x_{n+1} = \operatorname{argmin}_{y \in C} \left\{ \lambda F(y_n, y) + \frac{1}{2}\|y - x_{n+1}\|^2 \right\}, \end{cases}$$

where $\lambda > 0$.

The convergence of the algorithm is proved under the assumption that the solution exists and the bifunction is pseudo-monotone and Lipschitz-type.

References

1. Blum E., Oettli W. From optimization and variational inequalities to equilibrium problems // Math. Stud. 1994. V. 63. P. 123–145.
2. Combettes P.L., Hirstoaga S.A. Equilibrium programming in Hilbert spaces // Journal of Nonlinear and Convex Analysis. 2005. V. 6, № 1. P. 117–136.

3. Popov L.D. A modification of the Arrow-Hurwicz method for search of saddle points // Mathematical notes of the Academy of Sciences of the USSR. 1980. V. 28, № 5. P. 845–848.
4. Malitsky Yu.V., Semenov V.V. An Extragradient Algorithm for Monotone Variational Inequalities // Cybernetics and Systems Analysis. 2014. V. 50. P. 271–277.

Global optimality conditions for d.c. programming*

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Consider the optimization problem:

$$(\mathcal{P}): \qquad \left. \begin{array}{ll} f_0(x) \downarrow \min\limits_x, & x \in S \subset I\!R^n, \\ f_i(x) \leq 0, & i \in I := \{1, \ldots, m\}, \end{array} \right\} \tag{1}$$

where all $f_i = g_i(x) - h_i(x),\ \ i \in I \cup \{0\}$ with smooth convex functions $g_i(\cdot),\ \ h_i(\cdot),\ \ g_i, h_i:\ I\!R^n \to I\!R,\ i \in I \cup \{0\}$.

Let introduce the l_∞-penalty function [1]–[7]

$$W(x) := \max\{0, f_1(x), \ldots, f_m(x)\} = \max\{0, f_i(x), i \in I\}. \tag{2}$$

Further, consider the penalized problem as follows ($\sigma > 0$)

$$(\mathcal{P}_\sigma): \qquad \Theta_\sigma(x) := f_0(x) + \sigma W(x) \downarrow \min_x, \quad x \in S. \tag{3}$$

As well-known [1]–[7], if $z \in Sol(\mathcal{P}_\sigma)$, and $z \in D := \{x \in S:\ f_i(x) \leq 0,\ i \in I\}$, then $z \in Sol(\mathcal{P})$. In addition, if $z \in Sol(\mathcal{P})$, then under supplementary conditions [2, 3, 5, 7] for some $\sigma_* > 0$, $\sigma_* \geq \| \lambda_z \|_1$ (where λ_z is the KKT-multiplier corresponding to z), the inclusion $z \in Sol(\mathcal{P}_\sigma)$ holds. Moreover [6], $Sol(\mathcal{P}) = Sol(\mathcal{P}_\sigma)$, so that Problems $(\mathcal{P})$ and $(\mathcal{P}_\sigma)$ turn out to be equivalent $\forall \sigma \geq \sigma_*$.

It can be readily seen that the penalized function $\Theta_\sigma(\cdot)$ is a d.c. function, since the functions $f_i(\cdot),\ \ i \in I \cup \{0\}$, are as such. Actually, since $\sigma > 0$,

$$\Theta_\sigma(x) = G_\sigma(x) - H_\sigma(x), \tag{4}$$

*This research is supported by the Russian Science Foundation (grant 15-11-20015).

$$H_\sigma(x) := h_0(x) + \sigma \sum_{i\in I} h_i(x), \tag{5}$$

$$\begin{gathered} G_\sigma(x) := \Theta_\sigma(x) + H_\sigma(x) = \\ = g_0(x) + \sigma \max\left\{ \sum_{i=1}^{m} h_i(x); \max_{i\in I}[g_i(x) + \sum_{j\neq i} h_i(x)] \right\}, \end{gathered} \tag{6}$$

it is clear that $G_\sigma(\cdot)$ and $H_\sigma(\cdot)$ are convex functions.

For $z \in S$ denote $\zeta := \Theta_\sigma(z)$.

Theorem 1. *It $z \in Sol(\mathcal{P}_\sigma)$, then*

$$\forall (y,\beta) : \; H_\sigma(y) = \beta - \zeta, \tag{7}$$

the following inequality holds

$$G_\sigma(x) - \beta \geq \langle \nabla h_0(y) + \sigma \sum_{i\in I} \nabla h_i(y), x - y\rangle \quad \forall x \in S. \tag{8}$$

#

So, Theorem 1 reduces nonconvex (d.c.) Problem $(\mathcal{P}_\sigma)$ to a solving the family of convex linearized problems of the form

$$(\mathcal{P}_\sigma\mathcal{L}(y)) : \quad G_\sigma(x) - \langle \nabla H_\sigma(y), x\rangle \downarrow \min_x, \quad x \in S, \tag{9}$$

depending on the parameters (y,β) fulfilling the equation (7).

If for such a pair $(\hat{y},\hat{\beta})$ and some $u \in S$ (u may be a solution to $(\mathcal{P}_\sigma\mathcal{L}(y))$) the inequality (8) is violated, i.e.

$$G_\sigma(u) < \beta + \langle \nabla H_\sigma(y), u - y\rangle, \tag{10}$$

then due to convexity of $H_\sigma(\cdot)$ we obtain with the help of (7) that

$$G_\sigma(u) < \beta + H_\sigma(u) - H_\sigma(y) = H_\sigma(u) + \zeta.$$

The latter implies that $\Theta_\sigma(u) = G_\sigma(u) - H_\sigma(u) < \zeta := \Theta_\sigma(z)$, so that $u \in S$ is better that z, i.e. $z \notin (\mathcal{P}_\sigma)$.

It means that Global Optimality Conditions (7), (8) of Theorem 1 possesses the constructive (algorithmic) property allowing to construct local and global search methods for solving Problem $(\mathcal{P}_\sigma)$ [8, 9].

In particular, they enable us to escape a local pit of $(\mathcal{P}_\sigma)$ and to reach a global solution. The question arise about the existence of such a tuple (y,β,u). the answer is given by following result.

Theorem 2. *Let for a point $z \in S$ there exists $v \in I\!R^n$ such that*

$$(\mathcal{H}): \qquad \Theta_\sigma(v) > \Theta_\sigma(z).$$

If z not a solution to Problem $(\mathcal{P}_\sigma)$, then one can find a pair $(y, \beta) \in I\!R^{n+1}$, satisfying (7), and a point $u \in S$ such that the inequality (10) holds. #

Now let us set $y = z$ in (9). Then from (8) it follows that

$$\beta = \Theta_\sigma(z) + H_\sigma(z) = G_\sigma(z).$$

Furthermore, from (9) we derive

$$G_\sigma(x) - G_\sigma(z) \geq \langle \nabla H_\sigma(z), x - z \rangle \ \ x \in S,$$

that yields that z is a solution to the convex linearized problem

$$(\mathcal{P}_\sigma \mathcal{L}(z)): \quad G_\sigma(x) - \langle \nabla H_\sigma(z), x \rangle \downarrow \min_x, \quad x \in S,$$

As well-known [1]–[3], [6], due to the presentation (6) the latter problem amounts to the next one

$$\left.\begin{array}{c} g_0(x) - \langle \nabla H_\sigma(z), x \rangle + \sigma t \downarrow \min\limits_{(x,t)}, \quad x \in S, \ \ t \in I\!R, \\ \sum\limits_{i \in I} h_i(x) \leq t, \ \ g_i(x) + \sum\limits_{j \neq i} h_i(x) \leq t, \ \ i \in I. \end{array}\right\} \tag{11}$$

Moreover, the KKT-conditions to Problem (11) provide for KKT-conditions at z for the original Problem $(\mathcal{P})$.

So, the Global Optimality Conditions (7), (8) of Theorem 1 and 2 are connected with classical optimization theory [1]–[7].

References

1. Nocedal J., Wright S.J. Numerical Optimization. New York: Springer, 2006.
2. Bonnans J.-F., Gilbert J.C., Lemaréchal C., Sagastizábal C.A. Numerical Optimization: Theoretical and Practical Aspects. 2nd ed. Berlin: Springer-Verlag, 2006.
3. Izmailov A.F., Solodov M.V. Newton-Type Methods for Optimization and Variational Problems. New York: Springer, 2014.
4. Rockafellar R.T., Wets R.J.-.B. Variational Analysis. New York: Springer, 1998.
5. Clarke F.H. Optimization and Nonsmooth Analysis. New York: Wiley-Interscience, 1983.

6. Hiriart-Urruty J.-B., Lemaréchal C. Convex Analysis and Minimization Algorithms. Berlin: Springer-Verlag, 1993.
7. Burke J.V. An exact penalization viewpoint of constrained optimization // SIAM J. Control and Optimization. 1991. V. 29(4). P. 968-998.
8. Strekalovsky A.S. On Solving Optimization Problems with Hidden Nonconvex Structures // Optimization in Science and Engineering (ed. by T.M. Rassias, C.A. Floudas, S. Butenko). New York: Springer, 2014. P. 465–502.
9. Strekalovsky A.S. Elements of nonconvex optimization. Novosibirsk: Nauka, 2003 (in Russian).

Solving quadratic equation systems via nonconvex optimization methods*

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Consider the following system of quadratic equations [8]:

$$f_i(x) = \frac{1}{2}\langle x, C_i x\rangle + \langle d^i, x\rangle + \gamma_i = 0, \quad i = 1, 2, \ldots, m, \tag{1}$$

where C_i, $i = \overline{1,m}$, are, in general, indefinite $(n \times n)$-matrices such that

$$C_i = A_i - B_i, \quad A_i, B_i > 0 \quad \forall i \in \{1, 2, \ldots, m\}.$$

Further, we reduce system (1) to nonsmooth optimization problem as follows:

$$(\mathcal{P}): \quad F(x) = \sum_{i=1}^{m} |f_i(x)| = G(x) - H(x) \downarrow \min_x, \quad x \in I\!R^n, \tag{2}$$

where objective function $F(\cdot)$ is the (d.c.) function [1,2,6], which can be represented as a difference of two convex functions. For instance, we consider two d.c. representation $(j = 1, 2)$ of the form

$$F(x) = G_j(x) - H_j(x) \quad \forall x \in I\!R^n. \tag{3}$$

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Here the first d.c. representation (3) is given by the functions:

$$G_1(x) = 2\sum_{i=1}^{m} \max\{\tfrac{1}{2}\langle x, A_i x\rangle + \langle d^i, x\rangle + \gamma_i, \tfrac{1}{2}\langle x, B_i x\rangle\},$$
$$H_1(x) = \sum_{i=1}^{m} \Big[\tfrac{1}{2}\langle x, (A_i + B_i)x\rangle + \langle d^i, x\rangle + \gamma_i\Big].$$

Further, The second d.c. representation is as follows:

$$G_2(x) = \sum_{i=1}^{m} \max\{\langle x, A_i x\rangle + \langle d^i, x\rangle + \gamma_i, \langle x, B_i x\rangle - \langle d^i, x\rangle - \gamma_i\},$$
$$H_2(x) = \tfrac{1}{2}\sum_{i=1}^{m} \langle x, (A_i + B_i)x\rangle.$$

Note that in both d.c. representations (3) the functions $G_j(\cdot)$, $j = 1, 2$, are nonsmooth and functions $H_j(\cdot)$, $j = 1, 2$, are differentiable.

Proposition 1. *If z is a solution to problem $(\mathcal{P})$ and $F(z) = 0$, then z is a solution to system* (1).

For solving optimization problem $(\mathcal{P})$ we apply the Global Search Theory [1,2] based on necessary and sufficient global optimality conditions. Note that global search method includes two principal parts: local search and procedures of improving a critical point $z \in I\!R^n$ (i.e procedures for funding a point $u \in I\!R^n$ such that $F(u) < \zeta$, where $\zeta := F(z)$) provided by a local search method.

To this end for a fixed vector $y \in I\!R^n$ it is necessary to solve the following nonsmooth convex auxiliary (partially linearized) problem (both on every step of the special local search method and on the stage of improving a critical point):

$$(\mathcal{P}L(y)): \quad \Phi_y(x) = G_j(x) - \langle \nabla H_j(y), x\rangle \downarrow \min_x, \quad x \in I\!R^n, \quad j = 1, 2.$$

In order to perform it, we solve the nonsmooth problem $(\mathcal{P}L(y))$ via the smooth convex problem, increasing the dimension from n up to $(m + n)$. For the first case of d.c. representation (3) the problem $(\mathcal{P}L(y))$ is reduced to the following smooth convex optimization problem with quadratic inequality constraints:

$$\left\{\begin{array}{c} \theta_y(x,t) = \langle e, t\rangle - \langle \nabla H_1(y), x\rangle \downarrow \min\limits_{(x,t)}, \quad (x,t) \in I\!R^{n+m}, \\ \dfrac{1}{2}\langle x, A_i x\rangle + \langle d^i, x\rangle + \gamma_i \le \dfrac{t_i}{2}, \\ \langle x, B_i x\rangle \le t_i, \quad i = 1, 2, \ldots, m, \end{array}\right. \tag{4}$$

where $e = (1, 1, \ldots, 1)^\top \in I\!R^m$ and the gradient of $H_1(\cdot)$ at point $y \in I\!R^n$ is as follows

$$\nabla H_1(y) = \sum_{i=1}^{m}(A_i + B_i)y + \sum_{i=1}^{m} d^i.$$

In addition, for the second d.c. representation we employ another smooth convex optimization problem:

$$\left\{\begin{array}{c} \theta_y(x,t) = \langle e, t\rangle - \langle \nabla H(y), x\rangle \downarrow \min\limits_{(x,t)}, \quad (x,t) \in I\!R^{n+m}, \\ \langle x, A_i x\rangle + \langle d^i, x\rangle + \gamma_i \le t_i, \\ \langle x, B_i x\rangle - \langle d^i, x\rangle - \gamma_i \le t_i, \quad i = 1, 2, \ldots, m, \end{array}\right. \tag{5}$$

where

$$\nabla H_2(y) = \sum_{i=1}^{m}(A_i + B_i)y.$$

The computational experiments were carried out on test problems [9] with dimension up to 100. For solving smooth auxiliary problem (4) and (5) we apply existing methods and software (for instance, IBM ILOG CPLEX) for smooth convex optimization [3-5]. In addition, we compare the effectiveness of developed algorithms with rather popular solvers, for instance [7].

References

1. Strekalovsky A.S. Elements of Nonconvex Optimization. Novosibirsk: Nauka, 2003 (in Russian).
2. Strekalovsky A.S. On Solving Optimization Problems with Hidden Nonconvex Structures. In: Rassias, T.M., Floudas, C.A., Butenko, S. (eds.) Optimization in Science and Engineering. New York: Springer, 2014. P. 465–502.
3. Nocedal J., Wright S.J. Numerical Optimization. New York: Springer, 2006.
4. Bonnans J.-F., Gilbert J.C., Lemaréchal C., Sagastizábal C.A. Numerical Optimization: Theoretical and Practical Aspects, 2nd edn. Berlin, Heidelberg: Springer-Verlag, 2006.
5. Izmailov A.F., Solodov M.V. Newton-Type Methods for Optimization and Variational Problems. New York: Springer, 2014.
6. Hiriart-Urruty J.-B. Generalized Differentiability, Duality and Optimizaton for Problems dealing with Difference of Convex Functions. In: Ponstein, J. (ed.) Convexity and Duality in

Optimization. Lecture Notes in Economics and Mathem. Systems. V.256. Berlin: Springer-Verlag, 1985. P. 37–69.
7. Bellavia S., Macconi M., Morini B. STRSCNE: A Scaled Trust Region Solver for Constrained Nonlinear Equations // COAP. 2004. V. 28, №. 1. P. 31–50.
8. Ortega J.M., Rheinboldt W.C. Iterative Solution of Nonlinear Equations in Several Variables. New York: Academic Press, 1970.
9. Roose A., Kulla V., Lomp M., Meressoov T. Test examples of systems of non-linear equations. Tallin: Estonian Software and Computer Service Company, 1990.

Variant of simplex-like method for linear semi-definite programming problem*

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Let $\mathcal{S}^n$ denote the space of symmetric matrices of order n, and let $\mathcal{S}^n_+$ be the cone in $\mathcal{S}^n$, consisting of positive semi-definite matrices. We use also the inequality $M \succeq 0$ to indicate that a matrix M belongs to $\mathcal{S}^n_+$. The inner product of matrices M_1 and M_2 of the same size is defined as the trace of the matrix $M_1^T M_2$ and is denote by $M_1 \bullet M_2$.

The linear semi-definite programming problem is to find

$$\begin{gathered} \min\ C \bullet X, \\ A_i \bullet X = b^i, \quad i = 1, \ldots, m, \quad X \succeq 0, \end{gathered} \tag{1}$$

where the matrices $C \in \mathcal{S}^n$ and $A_i \in \mathcal{S}^n$, $1 \le i \le m$, are given. The matrix $X \in \mathcal{S}^n$ is a variable. We assume that the matrices A_i, $1 \le i \le m$, are linear independent.

The problem dual to (1) has the form

$$\begin{gathered} \max\ b^T u, \\ \sum_{i=1}^m u^i A_i + V = C, \quad V \succeq 0, \end{gathered} \tag{2}$$

where $b = [b^1, \ldots, b^m]$, $V \in \mathcal{S}^n$.

Let $n_\triangle = n(n+1)/2/$ be the n-th triangular number. Let also vech$X$ denote the direct sum of parts of columns of $X \in \mathcal{S}^n$ beginning with the diagonal entry. The dimension of vech$X$ is equal to $n_\triangle$. The operation

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svec$X$ is defined similarly. It differs from the preceding operation vech$X$ only in that the off-diagonal entries of X are multiplied by $\sqrt{2}$ before placing into svec$X$.

We denote by $\mathcal{L}_n$ and $\mathcal{D}_n$ the *elimination* and *duplicated* matrices respectively [1], and by $\tilde{\mathcal{L}}_n = D_2\mathcal{L}_n$, $\tilde{\mathcal{D}}_n = \mathcal{D}_n D_2^{-1}$. The matrix D_2 of order $n_\triangle$ is diagonal with the vector svec$E$ on its diagonal, where E is a matrix of ones.

The optimality conditions for both problems (1) and (2) can be written in vector form as

$$\begin{array}{rcl} \langle \mathrm{svec} X, \mathrm{svec} V \rangle & = & 0, \\ \mathcal{A}_{svec} \mathrm{svec} X & = & b, \\ \mathrm{svec} V & = & \mathrm{svec} C - \mathcal{A}_{svec}^T u, \end{array} \qquad (3)$$

where angle brackets indicate the Euclidean inner product in finite-dimensional vector space, and $\mathcal{A}_{svec}$ denotes the $m \times n^2$ matrix with svecA_i as its rows, $1 \le i \le m$. Matrices X and V must be positive semi-definite.

It is possible to obtain various numerical methods for problems (1) and (2), solving the system (3) by various ways. Here we consider the variant of simplex-like method.

Denote by $\mathcal{F}_P$ the feasible set in problem (1). Let $X \in \mathcal{F}_P$, and let

$$X = Q\mathrm{Diag}\left(\eta^1, \dots, \eta^r, 0, \dots, 0\right) Q^T,$$

where Q is an orthogonal matrix of order n, $\eta^i > 0$, $1 \le i \le r$. Let Q_B be the $n \times r$ matrix formed from the first r columns of Q, and let $A_i^{Q_B} = Q_B^T A_i Q_B$, $1 \le i \le m$. Then X is an extreme point of $\mathcal{F}_P$, if and only if

$$\mathrm{rank}\left[A_1^{Q_B}, \dots, A_m^{Q_B}\right] = r_\triangle.$$

Thus the point $X \in \mathcal{F}_P$ may be extreme only when the rank r of X is such that $r_\triangle \le m$. We say that the extreme point $X \in \mathcal{F}_P$ is *regular* if $r_\triangle = m$. Otherwise, in the case where $r_\triangle < m$, we call the extreme point X *irregular*.

Denote by $\mathcal{A}_{svec}^{Q_B}$ the $m \times r_\triangle$ matrix whose rows are vectors svecA_i, $1 \le i \le m$. Also denote by C^{Q_B} the matrix $Q_B^T C Q_B$ and by V^{Q_B} — the matrix $Q_B^T V Q_B$. It is evident that the first equality in (3) is fulfilled, if $V^{Q_B} = 0_{rr}$.

1. *Pivoting in a regular extreme point X*. In this case we have the system of linear equations

$$\mathrm{svec} V^{Q_B} = \mathrm{svec} C^{Q_B} - \left(\mathcal{A}_{svec}^{Q_B}\right)^T u = 0_{r_\triangle}, \qquad (4)$$

with the non-degenerate matrix $\mathcal{A}_{svec}^{Q_B}$ of order $m = r_\triangle$. Therefore

$$u = \left(\left(\mathcal{A}_{svec}^{Q_B} \right)^T \right)^{-1} \mathrm{svec} C^{Q_B}.$$

If the matrix $V(u) = C - \sum_{i=1}^m u^i A_i$ is positive semi-definite, then X is a solution of problem (1). In what follows we assume that it is not such a case.

Represent the matrix V in the form $V = H \mathrm{Diag}(\theta) H^T$, where H is an orthogonal matrix. Then there exists the eigenvalue $\theta^k < 0$ among all eigenvalues θ. Let h_k be the corresponding eigenvector. It can be proved that the vector h_k does non belong to the columns space of the matrix Q_B. The point X is updated in accordance with the following formulae

$$\bar{X} = X + \alpha \Delta X, \quad \Delta X = Q_B \Delta Z Q_B^T + h_k h_k^T, \tag{5}$$

where $\alpha > 0$ is a stepsize, and the matrix ΔZ satisfies to equations

$$A_i \bullet \left[Q_B \Delta Z Q_B^T + h_k h_k^T \right] = 0, \quad 1 \leq i \leq m.$$

The value of objective function $C \bullet X$ in the updated point $\bar{X}$ is less than in the previous point X, namely

$$C \bullet \bar{X} = C \bullet X + \alpha \theta_k < C \bullet X. \tag{6}$$

The point $\bar{X}$ is an extreme point of $\mathcal{F}_P$ too.

2. *Pivoting in an irregular extreme point* X. In this case the system (4) is underdetermined. Therefore we take the normal solution

$$u = \left(\mathcal{A}_{svec}^{Q_B} \right) \left[\left(\mathcal{A}_{svec}^{Q_B} \right)^T \left(\mathcal{A}_{svec}^{Q_B} \right) \right]^{-1} \mathrm{svec} C^{Q_B}.$$

The matrix ΔX in (5) is replaced by the following one

$$\Delta X = [Q_B \, h_k] \begin{bmatrix} \Delta Z & w \\ w^T & 1 \end{bmatrix} [Q_B \, h_k]^T,$$

where the vector w is chosen by a special way in order to preserve the formulae (6). Here we suppose in addition that $m = r_\triangle + p$ with $0 < p < r$.

Theorem. *Let the problem (1) be nondegenerate. Let also the starting extreme point* $X_0 \in \mathcal{F}_P$ *be such that the set*

$$\mathcal{F}_P(X_0) = \{ X \in \mathcal{F}_P : \; C \bullet X \leq C \bullet X_0 \}$$

is constrained. Then the sequences $\{X_k\}$ generated by the proposed method belongs to $\mathcal{F}_P(X_0)$ and converges to the solution of (1).

There are some other generalizations of simplex-method for linear semi-definite programming problems (see, for example, [2]).

References

1. Magnus J.R., Neudecker H. The elimination matrix: some lemmas and applications // SIAM J. Alg. Disc. Math. 1980. V. 1. № 4. P. 422–449.
2. Lasserre J.B. Linear programming with positive semi-definite matreces // MPE. 1996. V. 2. P. 499–522.

Covering constant of the restriction of a linear mapping to a convex cone*

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This work relies on the results in [1], and is an extension of the development in [2, 3].

Given a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^k$ and vectors $b_1, ..., b_s \in \mathbb{R}^n$, denote

$$K := \{x \in \mathbb{R}^n \,:\, \langle x, b_j \rangle \le 0, j = \overline{1, s}\}.$$

Here $\langle \cdot, \cdot \rangle$ states for inner product, $| \cdot |$ states for the corresponding Euclidian norm.

In this paper, we consider the problem of finding of a prior estimate for distance from an arbitrary point $x_0 \in K$ to the set of solutions to the system $Ax = y$, $x \in K$, where $y \in AK$ is an arbitrary point.

Hoffman's lemma implies that there exists $\alpha > 0$ such that

$$\forall\, x_0 \in K,\ \ \forall\, y \in AK\ \ \exists\, x \in K :\ \ y = Ax \ \text{ and } \ |x - x_0| \le \frac{|y - Ax_0|}{\alpha}. \quad (1)$$

So, the desired estimate is linear. There appears a natural question: how can the number α be calculated for given matrix A and vectors b_j. Below we state a proposition that reduces this problem to the same problem in the space $\mathbb{R}^{n-1}$ with the lower dimension.

The mentioned constant α is also called the covering constant of the mapping $A|_K : K \to AK$. Recall the corresponding concept. Let X, Y be metric spaces with metrics ρ_X and ρ_Y, respectively, $\alpha > 0$ be given.

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Definition. *The mapping* $\Psi : X \to Y$ *is called* α*-covering, if*

$$B_Y(\Psi(x_0), \alpha r) \subset \Psi\big(B_X(x_0, r)\big) \quad \forall x_0 \in X, \quad \forall r \geq 0. \tag{2}$$

The least upper bound of all positive α *for which (2) holds is called covering modulus of* Ψ*. We denote this number by* $\operatorname{cov}(\Psi)$*.*

The concept of covering was used in [1] to derive sufficient conditions for existence of coincidence points of two mappings. In [2], the stability of coincidence points of covering and Lipschitz mappings was proved. The covering mappings are applied for investigation of implicit ordinary differential equations (see [3]), abstract and integral Volterra equations (see [4]), implicit differential inclusions (see [5]), etc.

The stated definition directly implies that the mapping $A|_K : K \to AK$ is α-covering if and only if (1) holds. So, the initial problem can be stated as a the problem of finding of the mapping $A|_K$ covering constant. At the same time, the most interest causes not finding of $\alpha > 0$ satisfying (1), but the number $\operatorname{cov}(A|_K)$, since the interval $(0, \operatorname{cov}(A|_K))$ is the set of all the desired α.

Let us state the main result. Assume that

(i) interior of K is nonempty;

(ii) for each $j = \overline{1, s}$, inequality $\langle b_j, x\rangle \leq 0$ is not a consequence of the system $\langle b_i, x\rangle \leq 0$, $i \neq j$;

(iii) linear mapping A is not injective.

Denote by Γ_j the face of the cone K that is orthogonal to b_j, i.e. $\Gamma_j = \{x \in K : \langle b_j, x\rangle = 0\}$. It is a straightforward task to ensure that the dimension of Γ_j equals to $n-1$ if (ii) holds.

Lemma. *Assumptions (i)–(iii) implies* $\operatorname{cov}(A|_K) = \min\limits_{j=\overline{1,s}} \operatorname{cov}(A|_{\Gamma_j})$.

Assumptions (i)–(iii) are not burdensome. In order to assumption (ii) be satisfied, from the system $\langle b_i, x\rangle \leq 0$, $i = \overline{1, s}$, there can be excluded the inequalities $\langle b_j, x\rangle \leq 0$, which are consequences of the systems $\langle b_i, x\rangle \leq 0$, $i \neq j$. The set of solutions to the obtained system coincides with K. If the interior of K is empty then the initial problem can be considered on the linear hull of the cone K instead of $\mathbb{R}^n$. In this case assumption (iii) can be changed by the noninjectivity of A on the linear hull of K.

This lemma cannot be applied in the case when (i) and (ii) hold and (iii) is violated. However, in this case, it is obvious that $\operatorname{cov}(A|_K)$

coincide with cov(A), which is equal to the least eigenvalue of A^*A (see, for example, [6], §6.2.2).

References

1. Arutyunov A.V. Covering mappings in metric spaces and fixed points // Doklady Mathematics, 2007. V. 76. Iss. 2. P. 665–668.
2. Arutyunov A.V. Stability of Coincidence Points and Properties of Covering Mappings // Mat. Zametki, 2009. V. 86. Iss. 2. P. 163–169.
3. Arutyunov A.V., Avakov E.R., Zhukovskiy E.S. Covering mappings and their applications to differential equations unsolved for the derivative // Diff. Equations, 2009. V. 45. Iss. 5. P. 627–649.
4. Arutyunov A.V., Zhukovskiy E.S., Zhukovskiy S.E. Covering mappings and well-posedness of nonlinear Volterra equations // Nonlinear Analysis: Theory, Methods and Applications, 2012. V. 75. P. 1026–1044.
5. Arutyunov A.V., Pereira F.L., V.A. de Oliveira, Zhukovskiy E.S., Zhukovskiy S.E. On the solvability of implicit differential inclusions // Applicable Analysis, 2015. V. 94. Iss. 1. P. 129–143.
6. Ioffe A.D., Tikhomirov V.M. Theory of Extremal Problems. Moscow: Nauka, 1984 (in Russian).

Live migration of virtual resources in multi-tenant data centers *

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This work extends [1,2] and relies heavily on descriptions introduced in those works.

In a modern multi-tenant data centers scheduling is a crucial problem, that greatly affects performance and utilization of physical devices and overall computational capabilities. Heavily loaded data centers suffer from resource fragmentation and underutilization. These issues could be resolved during maintenance, but this always requires interruption of virtual resources accessibility, which is impossible in Infrastructure-as-a-Service (IaaS) model, where end user specifically demands uninterrupted services.

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In [1,2] we propose a basic mathematical definition of IaaS multitenant data center along with a tenant definition and a set of SLA constraints to describe uninterruptable service contracts. In [3] we define a scheduling algorithm that resolves placement problem of tenants onto physical resources while holding SLA's. This algorithm shows a significant improvement in resource utilization over those implemented in [4-6], providing either smaller load on each physical device, or full load on the subset of servers, allowing to shut down remaining ones thus increasing energy efficiency of data center.

The experimental research presented in [3] shows, that effective resource utilization in a heavily fragmented data center with more than 60% utilization yields many relocations of working virtual machines and database instances. The number of virtual resources relocated grows with the overall load of data center. To maintain uninterrupted service from those resources the data center control layer should provide mechanisms for live migration. A schedule of the live migration should be constructed by the scheduling algorithm that creates a tenant placement. If it can't devise the migration schedule for a given placement, the named placement should be rejected and reconstructed from scratch.

We describe a set of parameters that will affect migration of a virtual machine or a database instance in a live data center. The key parameters of migrating virtual machine in this respect are RAM consumption, RAM exchange speed and external communications speed. The parameter of data center is current load of it's resources, namely network resources. We claim that the time of migration of a given virtual machine depends only on those parameters. Sufficient network throughput between current working machine and it's mirrored replica on the destination server allows to transfer all working data to the destination virtual machine untill it is fully up to date with current virtual machine. The source machine can then be transparently disabled and removed from data center, thus finishing the migration process.

We then define a set of constraints that allows to calculate overall migration time of virtual machine on given data center workload and virtual machine parameters. This constraints allow to devise migration costs of all the virtual resources that need to be relocated alongside with a general feasibility of complete migration schedule.

Based on these constraints we introduce modified scheduler algorithm [3], that is aware of migration costs and is able to construct a feasible migration schedule. It allows to construct only resource placement that can be performed on a live data center without interrupting any of working or migrating virtual resources services. The algorithm can also be provided with a directive migration time, so that the constructed migration schedule does not exceed this additional constraint. Data center that utilizes the given scheduling algorithm is able to provide uninterrupted service as well as guarantee all tenants SLA's during the time of migration.

This work introduces mathematical apparatus to formulate and check time constraints for migration of virtual resources based only on their parameters and work load of data center. Using this apparatus we define a migration-aware scheduling algorithm that can be used as scheduler in data center, which implements IaaS model and is to provide uninterrupted service alongside with high utilization of it's resources.

References

1. P. M. Vdovin, I. A. Zotov, V. A. Kostenko, et al., "Data center resource allocation problem and approaches to its solution,"// in VII Moscow Int. Conf. on Operations Research (ORM2013) (Vychisl. Tsentr Ross. Akad. Nauk, Moscow, 2013), Vol. 2, pp. 30-32.
2. P. M. Vdovin, I. A. Zotov, V. A. Kostenko et al., "Comparing various approaches to resource allocating in data centers"// Journal of Computer and Systems Sciences International. – 2014. – Vol. 53, no. 5. – P. 689-701.
3. Zotov I. A., Kostenko V. A. "Resource allocation algorithm in data centers with a unified scheduler for different types of resources"// Journal of Computer and Systems Sciences International. – 2015. – Vol. 54, no. 1. – P. 59-68.
4. S. Nagendram, J. V. Lakshmi, D. V. Rao, et al., "Efficient resource scheduling in data centers using MRIS,"// Indian J. Comput. Sci. Eng. 2 (2011).
5. M. Korupolu, A. Singh, and B. Bamba, "Coupled placement in modern data centers,"// in IEEE Int. Symp. on Parallel & Distributed Processing (IPDPS, New York, 2009), pp. 1-12.
6. Y. Zhu and M. H. Ammar, "Algorithms for assigning substrate network resources to virtual network components,"// in 25th Int. Conf. on Computer Communications (INFOCOM), Barcelona, 2006, pp. 1-12.

Multiple objective decision making

Convolution methods for criteria of efficiency and risk in the problem of investment portfolio choice

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The development of optimality criteria for a securities portfolio involv- es solving the issue on the relationship between the return and risk of the portfolio. In [4], Markowitz stated the problem on the selection of an optimal portfolio as the problem of minimizing the difference between the variance and the expectation of the portfolio return. In addition, in the same book the problem of maximizing the expected return under a constraint on the variance is considered. The problem of minimizing the variance under the constraint on the return is also considered. Solutions of all these problems are efficient portfolios. In [2, 3], the problem on portfolio selection was considered as the problem of maximizing a linear convolution of criteria "expectation—variance" with a weight factor (risk coefficient). By the convexity of the set of attainable values for the expectation and variance of portfolios (in the "north-west" direction) it gives necessary and sufficient conditions for the Pareto optimality, i.e., any problem whose solution is an effective portfolio is equivalent to a given problem at a certain risk factor.

In [1], we considered the problem of minimizing the convolution of the ratio type with the risk function defined in the metric l_2 and the problem of minimizing the probabilistic risk function. We also proposed a method of reduction of such problems to problems of quadratic programming (for the problem of minimizing the probabilistic risk function under the assumption of a normal distribution of random returns of financial instruments).

Here we consider one of the possible statements, namely, we define an optimal portfolio as a solution of the problem of maximizing the expectation of the portfolio return, provided that the probability of a negative random value of the portfolio return does not exceed a given, sufficiently small value:

$$\max_x \bar{r}x, \quad P(rx \leq 0) \leq \varepsilon, xe = 1, \ x \geq 0, \tag{1}$$

where ε is a given sufficiently small positive value, $e = (1, ..., 1)$, and P is the probability, $\bar{r} = (\bar{r}_1, ..., \bar{r}_i, ..., \bar{r}_n)$ is the vector of expectations of financial instruments.

We show that problem (1) is reduced to a problem of convex programming and its solution coincides with the solution of the problem of maximizing the linear convolution of the criteria of the expectation and the standard deviation of the random portfolio return for some weight coefficient of the standard deviation. Consider the problem

$$\max_x \bar{r}x, \quad k\bar{r}x \geq (xKx)^{1/2}, xe = 1, \ x \geq 0, \tag{2}$$

for which the Lagrange function

$$L(x, \lambda) = \bar{r}x + \lambda(k\bar{r}x - (xKx)^{1/2}) \tag{3}$$

is defined on the set $X = \{x | xe = 1, \ x \geq 0\}$, λ is the Lagrange multiplier, k is a positive coefficient, $K = (\sigma_{ij})_{n \times n}$ is the covariance matrix.

Lemma. If the convex programming problem (2) has a solution x^0 and the corresponding Lagrange multiplier is positive, $\lambda^0 > 0$, i.e., (x^0, λ^0) is a saddle point of the function (3), then x^0 is a solution of the problem

$$\max_x [\bar{r}x - \frac{\lambda^0}{1 + \lambda^0 k}(xKx)^{1/2}], \quad xe = 1, \ x \geq 0. \tag{4}$$

Theorem 1. Let $\{r_i\}$ be a system of random variables each of which has a normal distribution, $\bar{r}_i$ be the expectation, $K = (\sigma_{ij})_{n \times n}$ be the covariance matrix, and let the conditions of the lemma hold. Then the solution of problem (1) coincides with the solution of the problem of maximizing the linear convolution of the criteria of the expectation and the standard deviation of the random portfolio return:

$$\max_{x\in X}[\bar{r}x-\alpha_1(xKx)^{1/2}], \tag{5}$$

where $\alpha_1=\frac{\lambda^0}{1+\lambda^0 d}$, $d=(\Phi^{-1}(1-2\varepsilon))^{-1}$, $\Phi(\cdot)$ is the Laplace function, λ^0 is the value of the Lagrange multiplier in problem (2).

Now we find an optimal portfolio as a solution of the problem of maximizing the linear convolution of the expectation and variance criteria for the portfolio return with the weight coefficient $\alpha>0$:

$$\max_{x}[\bar{r}x-\alpha(xKx)], \quad xe=1,\ x\geq 0. \tag{6}$$

We examine the following problem: In which case solutions of problems (1) and (6) coincide?

Theorem 2. Let x^0 be a solution of problem (1), the optimal value of the Lagrange multiplier in problem (2) is positive, $\lambda^0>0$, and the covariance matrix $K=(\sigma_{ij})_{n\times n}$ is strongly positive definite. Then there exists a value of the weight coefficient α in problem (6) such that the solutions of problems (1) and (6) coincide.

Theorem 2 proves the existence of a value of the risk coefficient α in problem (6) for which solutions of problems (1) and (6) coincide. However, Theorem 2 allows one to find the risk coefficient only by solving

$$\min_{x} xKx, \quad \bar{r}x\geq r_p^0,\ xe=1,\ x\geq 0, \tag{7}$$

where r_p^0 is the expected return of a portfolio at the solution point of problem (1), i.e. $\bar{r}x^0=r_p^0$. In the following assertion (Theorem 3), we obtain a value of the risk coefficient α for full-size portfolios.

Theorem 3. Let the conditions of Theorem 2 be satisfied and let a solution of problem (1) be a full-size portfolio. If in problem (6) the weight coefficient α satisfies the equation

$$4(1-\left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)^2 d^2)\alpha^2-4d^2\left(\frac{\bar{r}K^{-1}e}{eK^{-1}e}\right)\left(\bar{r}K^{-1}\bar{r}-\frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)\alpha+$$
$$+\left(\bar{r}K^{-1}\bar{r}-\frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)-d^2\left(\bar{r}K^{-1}\bar{r}-\frac{(eK^{-1}\bar{r})^2}{eK^{-1}e}\right)^2=0,$$

where $e=(1,...,1)$, $\bar{r}$ is the vector of expected returns, $d=(\Phi^{-1}(1-2\varepsilon))^{-1}$, $\varepsilon>0$, and $\Phi(\cdot)$ is the Laplace function, then solutions of problems (1) and (6) coincide.

Thus, if we use the model with a probabilistic risk function for the search for optimal portfolios, the results of this study make it possible to determine the equivalent ratio of the investor to risk. Problem (6) is computationally most convenient; for the search for an exact solution it is reduced to a system of linear equations. The results obtained in the present paper allow one to solve problem (6) instead of (1) for certain values of the parameters of these problems.

References

1. Gorelik V.A., Zolotova T.V. Criteria for evaluation and the optimality of risk in complex organizational systems. Moscow: CC RAS, 2009.
2. Gorelik V.A., Zolotova T.V. Some problems of the assessment of correlation of returns in investment portfolios // Probl. Upravl. 2011. в„– 3, P. 36–42.
3. Gorelik V.A., Zolotova T.V. Stability criteria for the stock market and their relationship with the awareness and the principles of investor behavior // Fin. Zh. 2013. в„– 3, P. 17–28.
4. Markowitz H. M. Portfolio Selection: Efficient Diversification of Investment. – N.-Y.: Wiley, 1959.

Composed methods to reduce the Pareto set*

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The paper deals with a multicriteria choice problem, which has in its setting a set of feasible alternatives X, a numerical vector criterion $f = (f_1, f_2, \ldots, f_m)$ and a binary preference relation $\succ_X$ of the Decision Maker which is defined on the set of alternatives and usually unknown. A set of selected alternatives is denoted by $C(X)$. This set is a solution of the multicriteria choice problem and one must be determined at the end of decision making process. Moreover, we introduce $C(Y) = f(C(X))$.

Let Y be a set of feasible vectors, i.e. $Y = \{y = f(x) \mid \text{for some } x \in X\}$. By $\succ_Y$ we shall denote a preference relation, defined on Y, and also

$$y \succ_Y y' \Longleftrightarrow x \succ_X x' \qquad \forall\, x \in \bar{x}, \forall x' \in \bar{x}',\ y = f(x),\ y' = f(x'),$$

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where $\bar{x}, \bar{x}'$ are classes of equivalence generated by the relation $x \sim x' \Leftrightarrow f(x) = f(x')$.

We shall assume that the following four reasonable axioms are fulfilled.

Axiom 1 (exclusion of dominated alternatives). *For any $y, y' \in Y$ the following implication $y \succ_Y y' \Rightarrow y' \notin C(Y)$ is true.*

Axiom 2 (transitivity). *There exists an extension $\succ$ of the relation $\succ_Y$ on all space R^m, and also $\succ$ is transitive.*

Axiom 3 (compatibility). *For $i = 1, 2, \ldots, m$ and for any two vectors $y, y' \in R^m$ such that*

$$y_k = y'_k, \forall k \neq i, \;\; y_i > y'_i$$

it follows that $y \succ y'$.

Axiom 4 (invariance with respect to linear positive transformation). *For any $y, y' \in R^m$ and arbitrary $\alpha > 0$, $c \in R^m$* the implication

$$y \succ y' \Rightarrow (\alpha y + c) \succ (\alpha y' + c)$$

is true.

We shall say [1] that *a quantum of information about the preference relation $\succ$* with two groups of criteria A, B and positive parameters w_i $(\forall i \in A), w_j$ $(\forall j \in B)$ is given if for any $y, y' \in R^m$ such that $y_i - y'_i = w_i$ $(\forall i \in A)$, $y'_j - y_j = w_j$ $(\forall j \in B), y_s = y'_s$ $(\forall s \notin (A \cup B)))$ the relation $y \succ y'$ holds.

Theorem 1.*Let $X \subset R^m$ be a convex set and a vector-function f be concave on it. Assume that we have a quantum of information about the preference relation with two groups of criteria A, B and the corresponding positive parameters. Then for any set of selected vectors $C(Y)$ the inclusion*

$$C(Y) \subset Closure(\bigcup_{\mu \in M} \{f(x^*) \in Y | \sum_{i=1}^{p} \mu_i g_i(x^*) = \max_{x \in X} \sum_{i=1}^{p} \mu_i g_i(x)\})$$

is true, where $M = \{\mu \in R^p | \mu_i > 0 \; \forall i, \;\; \sum_{i=1}^{p} \mu_i = 1\}$ and $p = m -$ card(B) + card(A)card(B) *components of g consist of $g_i = f_i \; \forall i \notin B$, $\;\; g_{ij} = w_j f_i + w_i f_j \; \forall i \in A$ and $\forall j \in B$.*

Theorem 2.*Let $X \subset R^m$ be a convex set and a vector-function f be bounded above and concave on it. Assume that we have a quantum of information about the preference relation with two groups of criteria A, B*

and the corresponding positive parameters. Then for any set of selected vectors $C(Y)$ the inclusion

$$C(Y) \subset Closure(\bigcup_{u \in U} \{f(x^*) \in Y | \|u - g(x^*)\| = \min_{x \in X} \|u - g(x)\|\})$$

is true, where $U = \{u \in R^p | u_i > \sup_{x \in X} g_i(x) \ for \ i = 1, 2, \dots, p\}$ and g is the same as in Theorem 1.

Remark 1. It must be noted that for polyhedral concave vector-function f and polyhedral set Y the operation 'Closure' may be omitted in above theorems.

Theorem 3. *Let f be an arbitrary m-dimensional numerical vector-function defined on arbitrary set X. Assume that $\alpha \in R$ and we have a quantum of information about the preference relation with two groups of criteria A, B and the corresponding positive parameters. Then for any set of selected vectors $C(Y)$ the inclusion*

$$C(Y) \subset \bigcup_{u \in U} \{f(x^*) \in Y | \max_{i=1,2,\dots,p} \|u_i - g_i(x^*)\| =$$

$$= \min_{x \in X} \max_{i=1,2,\dots,p} \|u_i - g_i(x)\|\}$$

is true, where $U = \{u \in R^p | \sum_{i=1,2,\dots,p} u_i = \alpha\}$ and g is the same as in Theorem 1.

Remark 2.All above results take place in general case when we have more than one quantum of information. Namely, a vector-function g may be obtained after taking into account not only one quantum but some finite collection of consistent information quanta about the preference relation. In this case the equality $p = m - \text{card}(B) + \text{card}(A)\text{card}(B)$ may be false. For more details about using an arbitrary collection of information quanta see [1-3].

References

1. Noghin V.D. Reducing of the Pareto Set: an Axiomatic Approach. Moscow: FIZMATLIT, 2016 (in Russian)
2. Noghin V.D., Baskov O.V. Pareto Set Reduction Based on an Arbitrary Finite Collection of Numerical Information on the Preference Relation // Doklady Mathematics, 2011. V. 83, No. 3. P. 418–420.
3. Noghin V.D. Reducing of the Pareto Set Algorithm Based on an Arbitrary Finite Set of Information "Quanta" // Scientific and Technical Information Processing, 2014. V. 41, No 5. P. 1-5.

Multicriteria optimization problem with dynamics*

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The linear optimal control problem with the fixed initial state and boundary condition in the form of a finite-dimensional multicriteria equilibrium problem is considered. This problem can be formulated in the following way [1]:

it is needed to find the control $u \in U$ and a vector $\lambda \in E^m_+$ satisfying the conditions:

$$\langle \lambda, f(x(t_1; u)) \rangle \to \inf, \tag{1}$$

$$\langle \mu - \lambda, f(x(t_1; u)) - \lambda \rangle \leqslant 0, \ \forall\, \mu \in E^m_+, \tag{2}$$

where

$$\dot{x}(t) = D(t)x(t) + B(t)u(t) + g(t), \quad t_0 \leqslant t \leqslant t_1, \quad x(t_0) = x_0, \tag{3}$$

Here $f(x) = (f^1(x), \ldots, f^m(x))$, $x \in E^n$ – is the given vector-function with convex, differentiable coordinates $f^i(x)$, $i = 1, \ldots, m$, $D(t)$, $B(t)$, $g(t)$ – matrices of corresponding sizes with piecewise continuous elements, t_0, t_1 — fixed time moments, $x_0 \in E^n$ — fixed initial state, $u = u(t) \in L^r_2[t_0, t_1]$ — control, $x = x(t; u) = (x^1(t), \ldots, x^n(t))$, $t_0 \leqslant t \leqslant t_1$ – system (3) trajectory, corresponded to the contol $u(t)$.

To find the solution of the problem (1)–(3) the extragradient method [2] is proposed and examined.

References

1. Antipin A.S., Khoroshilova E.V. Multicriteria boundary value problem in dynamics // Trudy Instituta Matematiki I Mekhaniki. 2015. V. 21, № 3. P. 20–29. (In Russian)
2. Vasiliev F.P. Optimization Methods. Moscow: MCCME, 2011.

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OR in economics

Threshold strategies in investor's behavior model*

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1. One of the fundamental problems of investing in real sector concerns the determination of optimal time for investment into a given project (see, e.g., classical monograph [1]).

The project is specified by the pair $(\pi_t, t \geq 0, I)$ where π_t is the revenue from the project at time t, and I means the amount of investment required to implement the project. Prices on input and output production are assumed to be stochastic, so π_t is considered as a stochastic process, defined at a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathrm{P})$.

At any time an investor can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information regarding its environment (prices of the product and resources, demand etc.). The goal of an investor in this situation is to find, using the available information, an optimal time for investing the project (investment timing problem), which maximizes the net present value from the project:

$$\mathrm{E}\left(\int_\tau^\infty \pi_s e^{-\rho s} ds - I e^{-\rho\tau}\right) \mathbf{1}_{\{\tau<\infty\}} \to \max_\tau, \tag{1}$$

where $\mathbf{1}_A$ is the indicator of A, and maximum is taken over all investment times τ.

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The majority of results on this problem (optimal investment strategy) has a threshold structure: to invest when present value from the project exceeds the certain level (threshold). In the heuristic level this is so for the cases of geometric Brownian motion, arithmetic Brownian motion, mean-reverting process and some other (see [1]). And the general question arises: For what underlying processes an optimal decision to an investment timing problem will have a threshold structure? Some sufficient conditions in this direction was obtained in [2].

If we denote $X_t = \mathrm{E}\left(\int_t^\infty \pi_s e^{-\rho(s-\tau)} ds \middle| \mathcal{F}_t\right)$ — present value of the project, implemented at the time t, then investment timing problem (1) can be viewed as a special case of optimal stopping problem:

$$\mathrm{E}^x \left(X_\tau - I\right) e^{-\rho\tau} \mathbf{1}_{\{\tau<\infty\}} \to \max_\tau,$$

where E^x means the expectation for the process X_t starting from the initial state x, and the maximum is considered over all stopping times τ.

Therefore, the question about a structure of optimal decision may be addressed to a general optimal stopping problem. Under what conditions (on both process and payoff function) an optimal stopping time will have a threshold structure? Some results in this direction (in the form of necessary and sufficient conditions) were obtained in [3,4] under some additional assumptions on underlying process and/or payoffs.

2. Let X_t be a diffusion process with values in the interval with boundary points l and r, where $-\infty \le l < r \le +\infty$, open or closed (i.e. it may be (l,r), $[l,r)$, $(l,r]$, or $[l,r]$), which is a solution to stochastic differential equation:

$$dX_t = a(X_t)dt + \sigma(X_t)dw_t, \quad X_0 = x,$$

where w_t is a standard Wiener process. Assume that $a(\cdot)$, $\sigma(\cdot)$ are continuous functions, and $\sigma(x) > 0$ for all $x \in (l,r)$. Under these assumptions the process X_t will be regular, i.e. starting from an arbitrary point x, the process reaches any point y in finite time with positive probability.

It is known that under the above assumptions there exist (unique up to constant positive multipliers) increasing and decreasing positive functions $\psi(x)$ and $\varphi(x)$, which are the fundamental solutions to the ODE

$$a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = \rho f(p) \qquad (2)$$

on the interval (l, r).

Let us define a threshold stopping time $\tau_p = \inf\{t \geq 0 : X_t \geq p\}$ — the first time when the process X_t *exceeds* level p.

Theorem 1. *Threshold stopping time τ_{p^*}, $p^* \in (l, r)$, is optimal in the investment timing problem* (1) *for all $x \in (l, r)$ if and only if the following conditions hold:*

$$(p - I)\psi(p^*) \leq (p^* - I)\psi(p) \quad \text{for } p < p^*; \tag{3}$$

$$\psi(p^*) = (p^* - I)\psi'(p^*);$$

$$a(p) \leq \rho(p - I) \quad \text{for } p > p^*,$$

where $\psi(x)$ is an increasing solution to ODE (2).

3. As Theorem 1 shows, under certain assumptions the optimal investment rule in problem (1) can be found over the class of all threshold stopping times $\{\tau_p,\ p \in (l, r)\}$. For this class the investment timing problem (1) can be written as follows:

$$(p - I)\,\mathrm{E}^x e^{-\rho\tau_p} \to \max_{p \in (l,r)}. \tag{4}$$

The following result gives necessary and sufficient conditions for optimal threshold.

Theorem 2. *Threshold $p^* \in (l, r)$ is optimal in the problem* (4) *for all $x \in (l, r)$, if and only if the conditions* (3) *and*

$$(p - I)/\psi(p) \quad \text{does not increase for } p \geq p^*,$$

hold, where $\psi(p)$ is an increasing solution to ODE (2).

References

1. Dixit A., Pindyck R.S. Investment under Uncertainty. Princeton: Princeton University Press, 1994.
2. Alvarez L.H.R. Reward functionals, salvage values, and optimal stopping // Math. Methods Oper. Res. 2001. V. 54, P. 315–337.
3. Arkin V.I. Threshold Strategies in Optimal Stopping Problem for One-Dimensional Diffusion Processes // Theory Probab. Appl. 2015. V. 59. P. 311–319.
4. Crocce F., Mordecki E. Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions // Stochastics. 2014. V. 86. P. 491–509.

The unified maximum principle for optimal economic growth problems*

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Let G be a nonempty open convex subset of $\mathbb{R}^n$ and let

$$f : [0,\infty) \times G \times \mathbb{R}^m \to \mathbb{R}^n \quad \text{and} \quad f^0 : [0,\infty) \times G \times \mathbb{R}^m \to \mathbb{R}^1.$$

The following problem (P) arise in many fields of economics, in particular in growth theory (see [1]):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty f^0(t,x(t),u(t))\,dt \to \max, \tag{1}$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0, \tag{2}$$

$$u(t) \in U(t). \tag{3}$$

Here $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the values of the state vector and the control vector at time $t \geq 0$, respectively, $x_0 \in G$ is the initial state and $U\colon [0,\infty) \rightrightarrows \mathbb{R}^m$ is a multivalued mapping with nonempty values.

Assume that for a.e. $t \in [0,\infty)$ the derivatives $f_x(t,x,u)$ and $f_x^0(t,x,u)$ exist for all $(x,u) \in G \times \mathbb{R}^m$, and the functions $f(\cdot,\cdot,\cdot)$, $f^0(\cdot,\cdot,\cdot)$, $f_x(\cdot,\cdot,\cdot)$, and $f_x^0(\cdot,\cdot,\cdot)$ are Lebesgue-Borel (LB) measurable in (t,u) for every $x \in G$, and continuous in x for almost every $t \in [0,\infty)$ and every $u \in \mathbb{R}^m$. The multivalued mapping $U(\cdot)$ is also assumed to be LB-measurable, i.e. the set $\operatorname{gr} U(\cdot) = \{(t,u) \in [0,\infty) \times \mathbb{R}^m \colon u \in U(t)\}$ is a LB-measurable subset in $[0,\infty) \times \mathbb{R}^m$.

By definition, $(x(\cdot),u(\cdot))$ is an *admissible pair* in problem (P) if $u(\cdot)$ is a Lebesgue measurable function satisfying (3) for all $t \geq 0$, $x(\cdot)$ is the corresponding to $u(\cdot)$ locally absolutely continuous solution of the Cauchy problem (2) on $[0,\infty)$ in G, and the function $t \mapsto f^0(t,x(t),u(t))$ is locally integrable on $[0,\infty)$. Thus, for an arbitrary admissible pair $(x(\cdot),u(\cdot))$ and any $T > 0$ the integral

$$J_T(x(\cdot),u(\cdot)) := \int_0^T f^0(t,x(t),u(t))\,dt$$

is well defined. An admissible pair $(x_*(\cdot),u_*(\cdot))$ is *optimal* in problem (P) if the corresponding improper integral in (1) converges (to a finite

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number) and the following inequality holds for any other admissible pair $(x(\cdot), u(\cdot))$:

$$J(x_*(\cdot), u_*(\cdot)) \geq \limsup_{T\to\infty} \int_0^T f^0(t, x(t), u(t))\, dt.$$

Following [3], we will impose the following conditions on admissible pairs $(x_*(\cdot), u_*(\cdot))$ in problem (P).

$(A1)$ *There exists a continuous function* $\gamma\colon [0,\infty) \mapsto (0,\infty)$ *and a locally integrable function* $\varphi\colon [0,\infty) \mapsto \mathbb{R}^1$ *such that* $\{x\colon \|x - x_*(t)\| \leq \gamma(t)\} \subset G$ *for all* $t \in [0,\infty)$ *and*

$$\max_{\{x\colon \|x - x_*(t)\| \leq \gamma(t)\}} \Big\{ \|f_x(t, x, u_*(t))\| + \|f_x^0(t, x, u_*(t))\| \Big\} \overset{a.e.}{\leq} \varphi(t).$$

$(A2)$ *There exists a number* $\beta > 0$ *and a nonnegative integrable function* $\lambda : [0,\infty) \mapsto \mathbb{R}^1$ *such that for all* $\zeta \in G$*, satisfying the inequality* $\|\zeta - x_0\| < \beta$*, the initial value problem* (2) *with* $u(\cdot) = u_*(\cdot)$ *and the initial condition* $x(0) = \zeta$ *(instead of* $x(0) = x_0$*) has a solution* $x(\zeta;\cdot)$ *on* $[0,\infty)$ *in* G *and*

$$\max_{x \in [x(\zeta;t), x_*(t)]} \Big| \langle f_x^0(t, x, u_*(t)), x(\zeta;t) - x_*(t)\rangle \Big| \overset{a.e.}{\leq} \|\zeta - x_0\| \lambda(t).$$

If $(x_*(\cdot), u_*(\cdot))$ is an admissible pair satisfying conditions $(A1)$ and $(A2)$ then the fundamental matrix solution $Z_*(\cdot)$ of the linear system

$$\dot{z}(t) = -f_x(t, x_*(t), u_*(t))\, z(t), \quad t \geq 0,$$

with initial condition $Z_*(0) = I$ where I is the identity matrix is well defined on $[0,\infty)$.

Let $(x_*(\cdot), u_*(\cdot))$ be an admissible pair that satisfies $(A1)$ and $(A2)$, and such that the functional (1) converges. Then without loss of generality one can assume that there is a neighborhood $\Omega \subset [0,\infty) \times G$ of the set $\operatorname{gr} x_*(\cdot) = \{(t, x_*(t))\colon t \geq 0\}$, such that for all $(t,\zeta) \in \Omega$ there is a solution $x(\zeta, t; \cdot)$ of the Cauchy problem

$$\dot{x}(s) = f(s, x(s), u_*(s)), \qquad x(t) = \zeta,$$

on $[0,\infty)$ in G, and for all $(t,\zeta) \in \Omega$ the integral

$$W(t,\zeta) = \int_t^\infty f^0(s, x(\zeta, t; s), u_*(s))\, ds$$

converges. Notice, that the meaning of $W(t,\zeta)$ is the conditional value of the capital stock ζ at time t under a given investment plan $u_*(\cdot)$ (see [2]).

Define the normal form Hamilton-Pontryagin function $\mathcal{H} : [0,\infty) \times G \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^1$ for problem (P) in the usual way:

$$\mathcal{H}(t,x,u,\psi) = f^0(t,x,u) + \langle f(t,x,u),\psi\rangle,\ t \geq 0,\ x \in G,\ u \in \mathbb{R}^m,\ \psi \in \mathbb{R}^n.$$

The following result unifies the normal form version of the Pontryagin maximum principle for problem (P) developed in [3] with the Hamilton-Jacobi-Bellman equation without any a priory regularity assumptions on the value function (see [2] for details).

Theorem 1. *Let $(x_*(\cdot),u_*(\cdot))$ is an optimal admissible pair in problem (P) that satisfies conditions $(A1)$ and $(A2)$. Then*

(i) the partial (Fréchet) derivative $W_x(t,x_*(t))$ exists for all $t \geq 0$, and

$$W_x(t,x_*(t)) = Z_*(t)\int_t^{\infty} Z_*^{-1}(s) f_x^0(s,x_*(s),u_*(s))\,ds, \qquad t \geq 0;$$

(ii) the partial derivative $W_t(t,x_*(t))$ exists for a.e. $t \geq 0$, and

$$W_t(t,x_*(t)) +$$
$$+ \sup_{u\in U(t)} \Big\{\langle W_x(t,x_*(t)), f(t,x_*(t),u)\rangle + f^0(t,x_*(t),u)\Big\} \overset{a.e.}{=} 0;$$

(iii) the vector function $t \mapsto \psi(t) = W_x(t,x_*(t))$, $t \geq 0$, is locally absolutely continuous and satisfies the core relations of the normal form maximum principle for problem (P):

$$\dot{\psi}(t) \overset{a.e.}{=} -\mathcal{H}_x(t,x_*(t),u_*(t),\psi(t)),$$
$$\mathcal{H}(t,x_*(t),u_*(t),\psi(t)) \overset{a.e.}{=} \sup_{u\in U(t)} \mathcal{H}(t,x_*(t),u,\psi(t)).$$

References

1. Acemoglu D. Introduction to modern economic growth, Princeton N.J.: Princeton Univ. Press, 2008.
2. Aseev S. M. Adjoint variables and intertemporal prices in infinite-horizon optimal control problems // Proceedings of the Steklov Institute of Mathematics, 2015. V. 290. P. 223–237.

3. Aseev S. M., Veliov V. M. Maximum principle for infinite-horizon optimal control problems under weak regularity assumptions // Proceedings of the Steklov Institute of Mathematics, 2015. V. 291. Suppl. 1. P. S22–S39.

Fresh look at fair division problems: case with a massive discrete component*

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One of very basic problems of mathematical economics is the question of fair distribution of various types of resources between agents with different subjective estimates of the resources. Typical examples are cake-cutting, chore-division or an apartment rent-partitioning. In the most general form the fair division means that in the result of the division the share of each agent is “not worse” than others. However, depending on the exact mathematical formalization of the word “worse”, the results (and even the existence of the solution) might be very different. There is a vast mathematical literature dedicated to these matters, see e.g. [1–7] and further references in these publications.

We introduce the notions of weak and strong solutions to the problem of fair division, generalizing the notions of “proportional” and “envy-free” notions used in the economics literature, and apply them for the analysis of the division of a resource having a massive discrete component, e.g. precious stones. Due to the complexity of the latter problem no approach to its solution exists in the literature.

Indeed, if the resource under division consists only of a number of stones of different prices there is no way to make a fair division. The situation changes if additionally there is a continuous component, e.g. some amount of money. Obviously this amount cannot be too small in order to make a change. We give necessary and sufficient conditions for the existence of weak and strong solutions for the fair division problem in terms of individual subjective estimates of the stones prices made by each of the agents and the total amount of money. The proof of this result is based on an explicit finite constructive algorithm of finding the solutions.

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An application of above mentioned ideas for the apartment rent-partitioning problem may be found in [8].

References

1. Steinhaus H. The problem of fair division // Econometrica, 16(1) (1948), P. 101–104.
2. Brams S.J., Michael A.J.,Klamler C. Better Ways to Cut a Cake // Notices of the American Mathematical Society 53 (11) (2006), P. 1314—1321.
3. Moulin H. Fair division and collective welfare. Cambridge: MIT Press, 2003.
4. Brams S.J. Mathematics and democracy: Designing better voting and fair division procedures. Princeton, N.J.: Princeton University Press, 2008.
5. Barbanel J., Brams S.J., and Stromquist W., Cutting a pie is not a piece of cake // American Mathematical Monthly 116 (2009), P. 496–514.
6. Su F.E. Rental Harmony: Sperner's Lemma in Fair Division // American Mathematical Monthly 106:10 (1999), P. 930—942.
7. Stromquist W. Envy free cake divisions cannot be found by finite protocols // Electronic Journal of Combinatorics 15(1) (2008), R11.
8. Blank M. Problem of a fair division of a hybrid resource // Problems of Information Transmission, (to appear).

The asymptotic solution of one problem of economic dynamics with turnpike properties of optimal trajectories*

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In this work the algorithm for the construction of approximated optimal solution of problems of economic dynamics where trajectories have turnpike character is proposed. At first the similar approach was described in [1]. This approach is based on the singular perturbations theory and allows to find zero uniform optimal control asymptotic approximations that lead to balanced growth trajectories for the economic

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growth model which combines the properties of dynamical models of Leontief and Neumann. Let us consider the discrete time dynamic models of the economic system [2] where the time step μ is a small parameter. The dynamic Leontief model of a multisector economy has the form

$$\begin{array}{c} x(t) = Ax(t) + B[x(t+\mu) - x(t)] + d(t), \;\; x(0) = x_0, \\ t \in T_\mu = \{t : t = k\mu, k = 0, 1, ..., (N-1), \, 0 < \mu << 1\} \end{array} \tag{1}$$

The von Neumann growth model may be presented as follows

$$x(t+\mu) = x(t) + (B_* - A_*)u(t), \;\; x(0) = x_0, \;\; t \in T_\mu, \tag{2}$$

If we combine models (1) and (2) and take the terminal criterion we get the following modified singular perturbed problem

$$P_\mu : J(u) = (x(T) - x_{fix})^T F(x(T) - x_{fix}) \to \min_u \tag{3}$$

$$x(t+\mu) = Ax(t) + (E+B)(B_* - A_*)u(t) + d(t), \, x(0) = x^0 \tag{4}$$

$$A_* u(t) \le x(t), \, u(t) \ge 0, \, x(t) \ge 0 \tag{5}$$

where x(t) – n-dimensional vector of output levels, $F = F^T > 0$, $A_{n \times n}$ is the Leontief input-output matrix, $B_{n \times n}$ is the matrix of capital coefficients, $u_i(t) \ge 0, u(t) = (u_1(t), ..., u_r(t))$ – production intensities vector during period $[t, \; t+\mu]$, $\; t \in T_\mu, \; j = \overline{1, r}$, $A_{*\, m \times r}$ and $B_{*\, m \times r}$ – nonnegative input and output matrices for the unit of production intensities, respectively, $(B_* - A_*)u(t)$ – net output vector at the end of the period $[t, \; t+\mu]$, $d(t) = \beta^t d(0)$ – vector of final demand, $\beta \ge 0$ – the balanced growth rate of consumption. System (4) can be interpreted as a dynamic balance equation, where the total output at the beginning of the next period must equal the sum of the consumption volume $d(t)$, the necessary investments in funds $B(B_* - A_*)u(t)$ required for the production of a selected amount of net output $(B_* - A_*)u(t)$ and the costs $Ax(t)$ of the technological processes functioning. The constraint (5) is taken from the von Neumann model. The criterion (3) is used to select admissible pairs $(x(t), u(t))$ that will ensure the best approximation to a certain specified target (x_{fix}) at the final time.

The proposed algorithm for the construction of zero uniform optimal control asymptotic approximations is based on the direct scheme of the boundary functions method [3,4], which is used to find the asymptotic approximation to the solution of problem (3)-(5) as the sum of the three series $z(t, \mu) \; = \; \bar{z}(t, \mu) + \Pi z(\tau_0, \mu) + Qz(\tau_1, \mu), \; z \; = \; \begin{pmatrix} x \\ u \end{pmatrix}$.

The series $\bar{z}(t,\mu)$ is the regular series with coefficients depending on t and $\Pi z(\tau_0\,,\mu)$, $Qz(\tau_1\,,\mu)$ - boundary layer series with the coefficients depending on $\tau_0 = \frac{t}{\mu}$, $\tau_1 = \frac{t-T}{\mu}$. It is assumed that terms of the boundary layer series have exponential estimates.

The steps of the algorithm are:

1) Substitute the power series expansion in the left and right hand sides of (3)-(5) and then equate the terms with the zero power of μ separately for the terms with t, τ_0, τ_1 to get three decomposition problems P_0, $\Pi_0\mathrm{P}$, $\mathrm{Q}_0\mathrm{P}$ for the identification of the zero terms of the control and state approximations.

2) From problem P_0 find the turnpike part of the trajectory $\bar{x}_0(t) = (E-A)^{-1}d(t)$ and the control function $\bar{u}_0(t) = c(t)u_A$, where u_A is the eigenvector of matrix $(B_* - A_*)$ corresponding to the zero eigenvalue, and $c(t)$ is an unknown scalar function. The following conditions must be satisfied $A_*\bar{u}_0(t) \leq \bar{x}_0(t)$, $\bar{u}_0(t) \geq 0$, $\bar{x}_0(t) \geq 0$.

3) Near the initial point we get the problem $\Pi_0\mathrm{P}$ as a system of inequalities for $\Pi_0 u(\tau_0)$ and $\Pi_0 x(\tau_0)$: $A_*(c(t)u_A + \Pi_0 u(\tau_0)) \leq (E-A)^{-1}d(t) + \Pi_0 x(\tau_0)$, $c(t)u_A \geq 0$, $(E-A)^{-1}d(t) + \Pi_0 x(\tau_0) \geq 0$, $c(t)u_A + \Pi_0 u(\tau_0) \geq 0$, $\Pi_0 x(\tau_0) = A^{\tau_0}\Pi_0 x(0) + \sum_{s=0}^{\tau_0 - 1} A^{\tau_0 - s - 1}(E+B)(B_* - A_*)\Pi_0 u(s)$, $\Pi_0 x(0) = x^0 - (E-A)^{-1}d(0)$.

4) Near the final point we have the next optimal control problem $J(u) = (\bar{x}_0(T) + Q_0 x(0) - x_{fix})^T F(\bar{x}_0(T) + Q_0 x(0) - x_{fix}) \to \min\limits_{Q_0 u(\tau_1), Q_0 x(0)} A_*(c(t)u_A + Q_0 u(\tau_1)) \leq (E-A)^{-1}d(t) + Q_0 x(\tau_1)$, $c(t)u_A + Q_0 u(\tau_1) \geq 0$, $(E-A)^{-1}d(t) + Q_0 x(\tau_1) \geq 0$,
$Q_0 x(\tau_1) = A^{\tau_1}(x_{fix} - \bar{x}_0(1)) - \sum_{s=\tau_1}^{-1} A^{\tau_1 - s - 1}(E+B)(B_* - A_*)Q_0 u(s)$.
It should be noted that due to the nature of the criterion (3), problems P_0 and $\Pi_0 P$ do not have a criterion, moreover, not for all elements of the solution of problems P_0, $\Pi_0 P$ and $Q_0 P$ a single value can be obtained.

5) Finally, as all of the described decomposition problems depend on one unknown discrete function $c(t)$, the following problem is solved
$J(u,t,\mu,c(t)) = (x(T,u_0(t,\mu,c(t))) - x_{fix})^T F(x(T,u_0(t,\mu,c(t))) - x_{fix}) \to \min\limits_{u_0(t,\mu,c(t))}$
$\bar{u}_0(t,c(t)) + \Pi_0 u(c(t),\frac{t}{\mu}) + Q_0 u(c(t),\frac{t-T}{\mu}) \geq 0$, $c(t) \geq 0$, $\forall t$,
$\bar{x}_0(t) + \Pi_0 x(\Pi_0 u,\frac{t}{\mu}) + Q_0 x(Q_0 u,\frac{t-T}{\mu}) \geq 0$, $A_* u_0(t,\mu,c(t)) \leq \bar{x}_0(t) + \Pi_0 x(\Pi_0 u,\frac{t}{\mu}) + Q_0 x(Q_0 u,\frac{t-T}{\mu})$.
For numerical calculations a small discrepancy functional (regularizator) can be additionally introduced in the criterion (3) to find the admissible controls.

6) If there exist $c(t)$ and $\Pi_0 u(c(t),\tau_0)$, $Q_0 u(c(t),\tau_1)$ such that the problems constraints are satisfied, we get optimal trajectory approximation $x(t,u_0(t,\mu,c(t)))$ from (4).

Thus, for the problem approximate solution the initial control problem (3)-(5) is reduced to the construction of the zero uniform optimal control asymptotic approximation. For sufficiently small μ the proposed algorithm gives a good approximation of the solution and requires less calculations in comparison with the direct solution of problem (3)-(5) as a discrete optimal control problem.

References

1. Danik Yu.E., Dmitriev M.G. Turnpike trajectories and singular perturbations // Trudy Instituta sistemnogo analiza RAN. 2012. V. 65, № 1. P. 60–67. (in Russian)
2. Al'sevich V.V. Vvedenie v matematicheskuju jekonomiku. Konstruktivnaja teorija: Uchebnoe posobie. Moscow: Knizhnyj dom «LIBROKOM», 2009. (in Russian)
3. Dmitriev M.G. Singular-perturbation theory and some optimal-control problems // Differential equations. 1985. V. 21, № 10. P. 1132–1136.
4. Vasil'eva A.B., Dmitriev M.G. Singular perturbations in optimal control problems // Journal of Soviet Mathematics. 1986. V. 34, № 3. P. 1579–1629.

Deterministic queuing system*

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Consider a deterministic queuing system which contains a single serving unit with three streams of applications. Speeds of receipt of applications as well as speeds of handling of applications by a service device depend on the quantity of the queue. At any moment the server can handle only one application. Service systems of such type have proliferated in recent years. For example, in various service centers an user of a terminal device chooses the queue number in accordance with the type of his application, then obtains a number in the chosen queue for service. The service comes with using a multifunctional operating device, which switches from one queue to another during an operation

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and wherein moments of switching are chosen by the service device. The formulated problem is similar to the well-known problem of the control of traffic lights at an isolated intersection [1] – [4], but significantly differs from it by the nature of the restrictions, in particular, it is generally assumed that the time of service in the problem of the intersection is equal to zero. Under the problem of managing such a system it is possible to understand the choice of the switching procedure of a servicing device with one queue to another, guaranteeing that there is no unlimited growth of the queue on each streams of applications. A similar problem with two streams was considered earlier in [4].

Introduce the following notation: Let $q_1(t)$, $q_2(t)$, $q_3(t)$ be queue lengths waiting for service of a multifunctional device for the first, second and third streams at the time t respectively. Let $a_i(t)$ и $d_i(t)$ be speeds of receipt and fulfillment orders for the i-th line, respectively, where $i = 1, 2, 3$. Let g_i be the duration of continuous service of requests from the queue with the number i, $g_i > 0$ $(i = 1, 2, 3)$

Let's assume that:

1) $a_i(t) = a_i \geq 0$ is a known constant;

2) $q_i(t)$ is a non-negative integer (the number of requests in the queue for service flow i at time t).

3)

$$d_i(t) = \begin{cases} 0, & \text{if the device supports an application from} \\ & \text{the queue } j \neq i\ ; \\ d_i, & \text{if the device supports an application from the queue } i\ ; \end{cases}$$

4) $d_i > a_i$, note that in the framework of our problem these quantities take constant values.

5) In the initial moment of time the queue is absent, i.e. $q_i(0) = 0, i = 1, 2, 3$.

6) Let the duration of continuous service requests from the same queue put the same for each of the queues.

Definition 1. *The triple* (g_1, g_2, g_3) *is called a cycle, where* g_i *is lengths of continuous service requests from the queue with the number i* $(i = 1, 2, 3)$

Let us consider three sequences of time points. The first sequence:

$$\tau_1^{(1)} = g_1,\ \tau_2^{(1)} = g_1 + g_2 + g_3 + g_1,\ \ldots,\ \tau_{k+1}^{(1)} = (g_1 + g_2 + g_3)k + g_1,\ \ldots.$$

This sequence of time points represents the points of start of service requests from the queue with the number two or the time of termination of the implementation of the requirements of the first stream.

The second sequence:

$$\tau_1^{(2)} = g_1 + g_2,\ \ \tau_2^{(2)} = g_1 + g_2 + g_3 + g_1 + g_2,\ \ \ldots,$$

$$\tau_{k+1}^{(2)} = (g_1 + g_2 + g_3)k + g_1 + g_2,\ \ \ldots.$$

The second sequence of time points represents the points of start of service requests from the queue with the number three, or the time of termination of the implementation of the requirements of the second stream.

The third sequence:

$$\tau_1^{(3)} = g_1 + g_2 + g_3,\ \ \tau_2^{(3)} = g_1 + g_2 + g_3 + g_1 + g_2 + g_3,\ \ \ldots,$$

$$\tau_{k+1}^{(3)} = (g_1 + g_2 + g_3)(k+1),\ \ \ldots.$$

The third sequence of time points represents points of start of service requests from the queue with the number one, or the time of termination of the implementation of the requirements of the third stream. Let's introduce a notation for the initial time: $\tau_0^{(0)} = 0$ is the start time of the MFD (a receiption of first request for service).

Definition 2. *Such regime of service applications in which there will be accumulation of the queue i.e., the following conditions*

$$q_1\left(\tau_{k+1}^{(1)}\right) = 0,\ q_2\left(\tau_{k+1}^{(2)}\right) = 0,\ q_3\left(\tau_{k+1}^{(3)}\right) = 0 \quad \forall k = 0, 1, 2, \ldots.$$

hold is called a stationary regime.

Now we find out conditions when the cycle (g_1, g_2, g_3) will lead to the stationary regime:

Theorem 1. *A cycle (g_1, g_2, g_3) generates a stationary regime if and only if when the following inequalities*

$$\frac{d_1 - a_1}{a_1} \geq \frac{g_2 + g_3}{g_1};\ \ \frac{d_2 - a_2}{a_2} \geq \frac{g_1 + g_3}{g_2};\ \ \frac{d_3 - a_3}{a_3} \geq \frac{g_1 + g_2}{g_3}$$

hold.

This theorem is proved similarly to the first Theorem from [4]. As opposed to Theorem from [4] in this theorem the question of the existence of a stationary regime for a service system with characteristics of d_i, a_i $(i = 1, 2, 3, \ldots)$ is not obvious. The following theorem gives an answer of this question.

Theorem 2. *Let $q_1(0) = q_2(0) = q_3(0) = 0$. The cycle (g_1, g_2, g_3) generating a stationary regime exists if and only if when the following conditions*

$$\frac{d_1 - a_1}{a_1} > \frac{a_2}{d_2 - a_2}; \quad \frac{d_3 - a_3}{a_3} > \frac{a_2}{d_2 - a_2};$$

$$\frac{d_3 - a_3}{a_3} > \frac{a_1}{d_1 - a_1}; \quad \frac{d_3}{a_3} \geq \frac{d_1 d_2}{d_1 d_2 - d_1 a_2 - d_2 a_1}$$

hold.

References

1. Aboudolas K., Papageorgiou M., Kosmatopoulos E., Store-and-forward based methods for the signal control problem in large-scale congested urban road networks // Transportation Research Part C: Emerging Technologies 2008. V. 17, № 2. P. 163–174
2. Diakaki C., Papageorgiou M., Aboudolas K. A multivariable regulator approach to traffic-responsive networkwide signal control. // Control Engineering Practice. 2002. № 10. P. 183–195.
3. Gazis D., Potts R. The oversaturated intersection // Proceedings of the International Symposium on the Theory of Traffic Flow. London: Elsevier. 1963. P. 221–227.
4. Haddad J., Mahalel D., Ioslovich I., Gutman P.-O. Constrained optimal steady-state control for isolated traffic intersections // Control Theory Tech. 2014. V. 12, № 1. P. 84–94.

Superhedging of American options in an incomplete $\{1, \bar{S}\}$-markets (discrete time, final horizon)

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There are many works devoted to the problem of American option's pricing in incomplete markets. For example, articles by Yu. M. Kabanov, V.I. Arkin, D. O. Kramkov, I. M. Sonin, A. N. Shiryaev, H. Föllmer, A. Schied, W. Schachermayer, F. Delbaen, R. Merton and other authors. There in the articles they have found conditions, when a solution exists for the problem in dynamic and static formulations. In the case of dynamic formulation this conditions are based on existence of uniform Doob decomposition (works by A. N. Shiryaev, H. Föllmer, A. Schied). In static case conditions of solution's existence for direct and "dual" variational problems were used (W. Schachermayer, F. Delbaen). But no methods to construct portfolio were proposed. This presentation differs from other works because we use minimax approach to solve the problem of American option pricing in an incomplete market in dynamic formulation. This approach enabled us to give a constructive description of superhedging portfolio and optimal exercise moment.

1. Formulation of the problem. Suppose, there is a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P})$ and d-dimensional $(d < \infty)$ adapted random sequence $\{S_n\}_{n\geq 0}$ on it. A market consisting of one risk-free asset with constant prise 1 and of d risky assets with prises evolving as $\{S_n\}_{n\geq 0}$ is called $\{1, \bar{S}\}$-*market* [1]. Let us denote: 1) $S_0^n \triangleq (S_0, ..., S_n)$, $n \geq 0$; 2) $N \in \mathbb{N}^+$ is a horizon; 3) $\mathfrak{R}_N \triangleq \{\mathsf{Q} : \mathsf{Q} \sim \mathsf{P}\}$; 4) $\mathfrak{M}_N \triangleq \{\mathsf{Q} : \mathsf{E}^{\mathsf{Q}}[S_{n+1}|\mathcal{F}_n] = S_n, n \geq 0\}$, where $\mathsf{E}^{\mathsf{Q}}[\cdot|\mathcal{F}_n]$ is the conditional expectation with respect to measure Q and σ-algebra $\mathcal{F}_n$. It is well known [1], that measure $\mathsf{Q} \in \mathfrak{R}_N$ specifies market and $\{1, \bar{S}\}$-market is incomplete if and only if $\mathfrak{R}_N \cap \mathfrak{M}_N \neq \varnothing$. We suppose that $\mathfrak{R}_N \cap \mathfrak{M}_N \neq \varnothing$ and there is no "friction" in $\{1, \bar{S}\}$-market.

Let $\mathcal{T}_n^N$ be a set of stopping moments τ taking values in the set $\{n, ..., N\}$ and $\{f_n\}_{0\leq n\leq N}$ is an adapted sequence of bounded random variables. *American option* is a contract between the Seller and the Buyer: 1) the Seller sales the right (the option) to buy from him or to sell him risky assets at any moment τ (chosen by the Buyer) at fixed conditions $\{f_n\}_{0\leq n\leq N}$ (dynamic payoff of American option); 2) the Buyer exercises option, i.e. the Seller buys or sells risky assets according the contract. To conclude the contract the Buyer pays the Seller fair value of the option. The Seller forms a portfolio of one risk-free and d risky assets $\pi \triangleq \{\beta, \gamma\}$ [1]. The set consisting of all $\gamma_1^N \triangleq (\gamma_1, ..., \gamma_N)$ will be denoted by D_1^N. The restriction of the set D_1^N to the set $\{n, ..., N\}$ we denote by D_n^N.

We treat the problem of American option pricing in an incomplete $\{1, \bar{S}\}$-market [1] as a stochastic game between the Seller and the Buyer. The Seller has portfolios π as his strategies. Exercise moments $\tau \in \mathcal{T}_n^N$ are Buyer's strategies. Suppose, that Seller's risk function is exponential and depends on deficit of his or her portfolio's capital. So exponential expected risk of the Seller at a moment $n \in \{0, ..., N\}$ with respect to any $\mathsf{Q} \in \mathfrak{R}_N$ is represented by the following formula

$$\begin{cases} I^{(\mathsf{Q},\tau),\gamma_{n+1}^N}(n, S_0^n) \triangleq \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{f_{(n\vee\tau)\wedge N} - \sum_{i=n+1}^{\tau\wedge N}(\gamma_i, \triangle S_i)\right\}|\mathcal{F}_n\right], \\ I^{(\mathsf{Q},\tau)}(N, S_0^N) \triangleq \exp\{f_N\}. \end{cases}$$

Let $\hat{D}_n^N \triangleq \left\{\gamma_n^N \in D_n^N : \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_n^N, \mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau),\gamma_n^N}(n-1, S_0^{n-1}) < \infty \ \mathsf{P}\text{-a.s.}\right\}$. Obviously, $\hat{D}_n \neq \varnothing$, $1 \leq n \leq N$.

Suppose, that neither the Seller, nor the Buyer knows risk assets prices' distribution $\mathsf{Q} \in \mathfrak{R}_N$. The Seller have to fulfill his obligation according to an option for sure. Also we suppose, that the Seller is rational, i.e.: 1) he or she assumes, that distribution of risk assets' prices and exercise moment (chosen by the Buyer) will maximaze his or her expected risk; 2) he or she managers portfolio to minimazes own expected risk. Thus, the Seller have to solve the following minimax problem:

$$I^{(\mathsf{Q},\tau),\gamma_1^N}(0, S_0) \to \operatorname*{ess\,inf}_{\gamma_1^N\in\hat{D}_1^N} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N}.$$

2. Important results.

Let $v_n^N \triangleq \operatorname*{ess\,inf}_{\gamma_{n+1}^N\in\hat{D}_{n+1}^N} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_n^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau),\gamma_{n+1}^N}(n, S_0^n)$ be the *upper guaranteed value of Seller's expected risk* at a moment $n \in \{0, ..., N\}$.

Theorem 1. *Suppose $\{S_n\}_{n\geq 0}$ is a d-dimensional adapted random sequence, $\{f_n\}_{0\leq n\leq N}$ is an adapted random sequence of bounded random variables and $\mathfrak{R}_N \triangleq \{\mathsf{Q} : \mathsf{Q} \sim \mathsf{P}\}$. Then the sequence $\{v_n^N\}_{0\leq n\leq N}$ satisfies the following recurrent relation* P*-a.s.*

$$\begin{cases} v_n^N = \max\left\{e^{f_n};\ \operatorname*{ess\,inf}_{\gamma_{n+1}\in\hat{D}_{n+1}} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\left[v_{n+1}^N e^{-(\gamma_{n+1},\, \triangle S_{n+1})}|\mathcal{F}_n\right]\right\}, \\ v_n^N|_{n=N} = e^{f_N}. \end{cases}$$

Theorem 2. *Suppose conditions of Theorem 1 are satisfied and $\mathfrak{R}_N \cap \mathfrak{M}_N \neq \varnothing$. Than for any $n \in \{1, ..., N\}$ there is $\gamma_n^* \in \hat{D}_n$ such, that* P*-a.s.*

$$\begin{aligned} &\operatorname*{ess\,inf}_{\gamma_n\in\hat{D}_n} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\left[v_n^N e^{-(\gamma_n,\, \triangle S_n)}|\mathcal{F}_{n-1}\right] = \\ &\qquad = \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} \mathsf{E}^{\mathsf{Q}}\left[v_n^N e^{-(\gamma_n^*,\, \triangle S_n)}|\mathcal{F}_{n-1}\right]. \end{aligned} \tag{1}$$

Remark. Suppose for any $n \in \{1, ..., N\}$ there is $\gamma_n^* \in \hat{D}_n$ such, that (1) is true P-a.s. As set $\mathcal{T}_0^N$ is finite, there always exists $\tau^* \in \mathcal{T}_0^N$: $\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau),\gamma_1^{*N}}(0, S_0) = \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau^*),\gamma_1^{*N}}(0, S_0)$.

Theorem 3. *Suppose conditions of Theorem 1 are satisfied and there are $\left(\gamma_1^{*N}, \tau^*\right) \in \hat{D}_1^N \times \mathcal{T}_0^N$ such, that* P*-a.s.*

$$\operatorname*{ess\,inf}_{\gamma_1^N\in\hat{D}_1^N} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N} \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau),\gamma_1^N}(0, S_0) = \operatorname*{ess\,sup}_{\mathsf{Q}\in\mathfrak{R}_N} I^{(\mathsf{Q},\tau^*),\gamma_1^{*N}}(0, S_0). \tag{2}$$

Then there exists a non-decreasing sequence $\{C_n^*\}_{0\le n\le N}$ *such, that* $f_{\tau^*\wedge N} = \ln v_0^N + \sum\limits_{i=1}^{\tau^*\wedge N}(\gamma_i^*, \triangle S_i) - C^*_{\tau^*\wedge N}$, $C_0^* = 0$ P-*a.s.*

Obviously, for any $n \in \{1, ..., N\}$ there is β_n^*: $\triangle\beta_n^* \triangleq -(S_{n-1}, \triangle\gamma_n^*)$, $\beta_0^* = \ln v_0^N$. A pair $\{\pi^*, C^*\}$ is called *superhedging portfolio* [1]. In [2] it is proved that capital of the superhedging portfolio $\{\pi^*, C^*\}$ is minimal among capitals of all other superhedging portfolios. This justifies our choice of exponential utility.

2.4. **Corollary 4.** *Suppose a stopping moment* $\tau^* \in \mathcal{T}_0^N$ *satisfies (2). Then it is possible to represent* $\tau^* \in \mathcal{T}_0^N$ *by the formula:*

$$\tau^* = \min\left\{0 \le n \le N : v_n^N = \exp\{f_n\}\right\}.$$

References

1. Föllmer H., Schied A. Stochastic Finance. An Introduction in Discrete Time. Berlin: Walter de Gruyter, 2004.
2. Khametov V.M., Shelemekh E.A. Superhedging of American options on an incomplete market with discrete time and finite horizon // Automation and Remote Control. 2015. 76:9. P. 1616–1634.

Analysis of 2015 Chinese stock market crash by means of generalized nonparametric method*

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The generalized nonparametric method [1–3] is based on the results of revealed preference theory which is devoted to solving the inverse problem of the demand analysis. The direct problem of the demand analysis is: given a utility function F, a price vector P and a level of expenditure I to find the optimal demand vector by solving the following problem: $\max\limits_{X\geqslant 0} F(X)$, $\langle P, X\rangle \leqslant I$.

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The inverse problem (for a finite number of observations) is: given a set $\{(P^t, X^t)\}_{t=1}^T$ of observed prices P^t and consumption vectors X^t (we call this set a trade statistics) to find a utility function F which rationalizes the observed data, that is, each X^t solves $\max\limits_{X \geqslant 0} F(X)$, $\langle P^t, X\rangle \leqslant \langle P^t, X^t\rangle$.

When solving the inverse problem one put several requirements on the utility function F. In the nonparametric method for market analysis and its generalized counterpart we put an additional requirement of positive-homogeneity of utility function (see [2], [3] for more details).

The inverse problem not always has a solution. When it does not, we introduce the irrationality index, which shows the degree violation of the existence conditions and come to the generalized nonparametric method for market analysis. The method allows one to compute economic indices and predict demand for an arbitrary price vectors.

The irrationality index [4] may be defined as the optimal value ω^* of the goal function in the following linear program: $\min \omega$, $\omega + \lambda_t - \lambda_\tau \geqslant c_{t\tau}$, $(t, \tau = 1, \dots, T)$, $\omega \geqslant 0$, where $c_{t\tau} = \log\left(\frac{\langle P^\tau, X^\tau\rangle}{\langle P^t, X^\tau\rangle}\right)$. If the irrationality index is zero, then the inverse problem has a solution.

The generalized nonparametric method allows one to make predictions about consumption at an arbitrary price vector. Suppose we have a trade statistics $\{(P^t, X^t)\}_{t=1}^T$ with irrationality index ω and a price vector P. Then the set of predicted volumes $K(P)$ is defined as the set of all nonnegative X such that the joint trade statistics

$$\{(P^t, X^t)\}_{t=1}^T \cup \{(P, X)\}$$

has the irrationality index ω.

One may show (see [5]) that if $e^\omega \geqslant 1$, then

$$K(P) = \{X \geqslant 0 \mid \gamma_\tau(P)\langle P^\tau, X\rangle \geqslant \langle P, X\rangle, \tau = \overline{1, T}\},$$

where

$$\gamma_\tau(P) = \min_{t \in \{1,\dots,T\}} \left\{ \frac{\omega^2}{C^*_{t\tau}} \frac{\langle P, X^t\rangle}{\langle P^t, X^t\rangle} \right\},$$

and

$$C^*_{t\tau} = \max\{\omega^{-k-1} C_{tt_1} C_{t_1t_2} \dots C_{t_{k-1}t_k} C_{t_k\tau} \mid \{t_1, \dots, t_k\} \subset \{1, \dots, T\},\ k \geqslant 0\},$$

The values $C^*_{t\tau}$ may be effectively computed in $O(T^3)$ operations by means of Floyd-Warshall algorithm.

We present a new methodology for analyzing stock markets based on generalized nonparametric method. We use a linear program from [6]

$$\sum_{t=1}^{T}\sum_{\substack{\tau=1\\ \tau\neq t}}^{T}\omega_{t\tau}\to \min \tag{1}$$

$$\omega_{t\tau}+\lambda_t-\lambda_\tau \geqslant c_{t\tau}-\omega_{min}, \qquad (t,\tau=\overline{1,T}), \tag{2}$$

$$\omega_{t\tau}\geqslant 0. \qquad (t,\tau=\overline{1,T}), \tag{3}$$

Here $\omega_{min}\geqslant 0$ is the allowed level of irrationality.

The dual problem is

$$\sum_{t=1}^{T}\sum_{\substack{\tau=1\\ \tau\neq t}}^{T}(c_{t\tau}-\omega_{min})x_{t\tau}\to \max \tag{4}$$

$$0\leqslant x_{t\tau}\leqslant 1, \qquad (t,\tau=\overline{1,T}) \tag{5}$$

$$\sum_{\tau=1}^{T}x_{t\tau}=\sum_{\tau=1}^{T}x_{\tau t}. \qquad (t=\overline{1,T}) \tag{6}$$

If $\{x^*_{t\tau}\mid t,\tau=1,\dots,T\}$, solve (4)–(6), then $x^*_{t\tau}\in\{0,1\}$. This property allows one to visualize the solution of the dual problem as a directed graph.

The solution of each of the problems (1)–(3) and (4)–(6) allows one to select the most irrational pairs of periods. This reduces markedly the number of periods an analyst needs to study carefully when analyzing some event on the financial markets. Then we use nonparametric predictions to analyze particular stocks that might cause the crash.

In this talk we present our results of applying this new methodology for analyzing the crash of Chinese stock market in 2015.

References

1. Afriat S.N. On a system of inequalities in demand analysis: an extension of the classical method. // International economic review. 1973. V. 14, № 2. P. 460–472.
2. Shananin A.A. Integrability problem and the generalized nonparametric method for the consumer demand analysis (Russian). // Proceedings of MIPT. 2009. V. 1, № 4. P. 84–98.

3. Klemashev N.I., Shananin A.A. Inverse problems of demand analysis and their applications to computation of positively-homogeneous Konüs–Divisia indices and forecasting. // Journal of Inverse and Ill-posed Problems. 2015. Advance online publication. DOI: 10.1515/jiip-2015-0015.
4. Grebennikov V.A., Shananin A.A. Generalized nonparametrical method: Law of demand in problems of forecasting. // Mathematical Models and Computer Simulations. 2009. V.1, № 5. P. 591–604.
5. Shananin A.A., Tarasov S. Computing the class of the form of the inverse demand function on discrete data. 58 MIPT conference. 2015.

System dynamic credit risk model of the corporate borrower

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Nowadays system dynamics is often used for solving various economic and social problems. System dynamics offers an approach in which the model resembles reality structurally, so we can validate it's usefulness and consistency. Furthermore, it offers a way to see the ramifications of that simplification through simulation, so we can test our hypotheses. System dynamics [1, 2] is a perspective and set of conceptual tools that enable us to comprehend the structure and dynamics of complex systems. System dynamics is also a rigorous modeling method that enables us to perform formal computer simulations of complex systems and use them for different purposes. This approach to understanding the nonlinear behavior of complex systems over time uses specialized concepts, which are the elements of any system dynamics model: stocks, flows, internal feedback loops, and time delays. Each of these elements is interpreted in different ways. Mathematically, the basic structure of a formal system dynamics computer simulation model is a system of coupled, nonlinear, first-order differential (or integral) equations.

This work focuses on the development of a system dynamic credit risk model of the company "Bashneft", which is a major representative of petroleum refining and petroleum producing industries.

The author intends to explore the possibility of using system dynamics to build models describing production process and financial conditions for a company. Special attention is paid to how the behavior of different macroeconomic factors influences the oil corporation. It's worth noting that the crude oil prices and oil product prices (on global and Russian markets) are among the most significant factors. In this case, the author considers such petroleum products as fuel oil, diesel fuel, and gasoline. Apart from those factors, US dollar rate and tax system (mineral extraction tax, export duties, petroleum products domestic excise tax) have a direct effect on the stability of the model. In addition, MosPrime rate is an important macroeconomic factor and a component of various structures of the system dynamics model. Moscow Prime Offered Rate is a reference rate fixed by the National Foreign Exchange Association (NFEA) based on the offer rates of Russian ruble deposits as quoted by contributor banks - the leading participants of the Russian money market to the first class financial institutions.

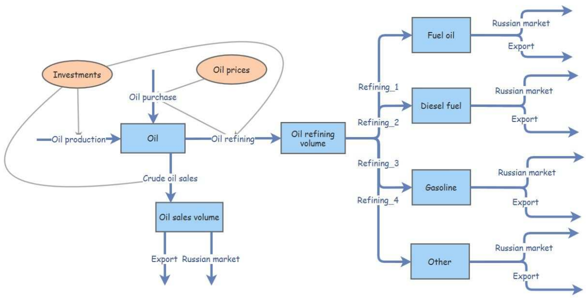

Fig. 1. Stock and flow representation of a manufacturing process.

At the beginning of this work, a detailed analysis of the oil company quarterly financial statements for the last 5 years was conducted. It allowed to identify the component parts of the model and to formalize some relationships between them. Then system dynamics tools were employed to observe how these relationships influence the behavior of the system over time. The result was a model that captures not only the current state of the company, but also the further development of its policy. This behavior is adjusted by changing external macroeconomic factors (implemented direct links) and controlled by the interaction of internal factors, realized by direct links and feedback loops. Internal factors may include oil production volume, oil refining volume, different types of costs, loan policy, and the volume of investments. Investments are aimed at reducing the cost of petroleum refining and petroleum production. The obtained model can be divided into two global parts that interact with each other. The first part describes the production process. It determines the volume of oil production, purchase and processing and the influence on the company profit. The second part is related to the financial unit of the company. This determines the level of debt, costs and loan for the considered corporation. As a result, the model allows to understand the strategy, level of loss and the probability of default for the company in the presence of various macroeconomic factors.

References

1. Sterman J.D. Business Dynamics: Systems Thinking and Modeling for a Complex World. Boston: McGraw-Hill Companies, 2000.
2. Katalevsky D.U. Fundamentals Of Simulation Modeling And System Analisys. Moscow: Moscow University Press, 2011.

Gender influences on the participants behavior in the economic experiments*

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MIPT Laboratory of experimental economics has been carrying the experiments beginning Fall 2013 [1, 2]. The goal of that is to study cooperation in social dilemmas. Each experiment consistes of a different set of 12 people, pre-selected before the experiment to be unfamiliar with one another. All participants are pre-tested using psychological questionaries.

The first step in every experiment begins from anonymous game phase, where participants played 2x2 economic games. Participants are randomly paired with an anonymous partner each period of the game. Number of periods is not known to participants. Each period participants are given information only about their profit for that period. After that, we carry the initial step of group socialization: in a sequence the participants tell their names and adjectives that start from the same letter, in a reverse order share their life facts, and divide into the groups. Finally, the participants play the same games like in the first step in the newly formed groups during the socialization.

There is two series of the experiments:

1.

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In the first step two games Prisoners' Dilemma and Ultimatum Game are conducted. After the socialization phase two people from participants voluntarily become leaders. The other participants decide one by one which leader they want to join. Thus two groups of 6 are formed. Both groups are asked to performe some group task. Series 1 consisted of 27 experiments (N=324, 202 males)[3].

2.

Unlike in series 1, in series 2 we use Prisoners' Dilemma and Trust Game. Participants divide into the groups this way: two people from participants voluntarily become leaders; players that are not leaders are asked to decide which a leader they want to join. On a piece of paper they indicate their choice of leader and how much money they are willing to pay for joining the group. After that we form 3 groups of 4 people. Two groups that include leaders have to perform some group tasks. Participants from the last group without a leader are not able to speak to or even to look at each. Therefore, the last group is not socialized. Series 2 consistes of 5 experiments (N=60, 45 males).

Results:

1. Socialization influences decisions in Prisoners' Dilemma and Ultimatum Game in different ways for males and females .

In Prisoners' Dilemma the initial (before socialization) level cooperation among women is higher than among men in series 1 (on average $\delta = .02$, $N_m = 202, N_f = 122$, wilcoxon-test, p-value = .05) [4], in series 2 (on average $\delta = .12$, $N_m = 45, N_f = 15$, wilcoxon-test, p-value = .05). Whereas after the socialization the percentage of choosing cooperative strategies among males in series 1 increases (on average $\delta = .35$, $N_m = 202$, wilcoxon-test, p-value $< 0{,}001$), in series 2 (on average $\delta = .53$, $N_m = 45$, wilcoxon-test, p-value $< 0{,}001$). Among females the percentage of choosing cooperative strategies in series 1 increases (on average $\delta = .18$, $N_f = 122$, wilcoxon-test, p-value $< .001$), in series 2 (on average $\delta = .42$, $N_f = 15$, wilcoxon-test, p-value $< .001$).

In Ultimatum Game the initial levels of cooperation for males and females are equal. However, after the socialization the level of cooperation for males is higher than for females (on average $\delta = .2$, $N_m = 202, Nf_f = 122$, wilcoxon-test, p-value=0,04).

2. In Trust Game males trust less than females, but reciprocate more.

In Trust Game we analyzed the "average trust"and "the average gratitude". Before socialization males trust less than females (on average $\delta = .58$ $N_m = 45, N_f = 15$, wilcoxon-test, p-value = .07) and they reciprocate more (on average $\delta = 1.18$, $N_m = 45, N_f = 15$, wilcoxon-test, p-value = .14). After socialization males are less trust than females (on average $\delta = .76$, $N_m = 45, N_f = 15$, wilcoxon-test, p-value = .03) and they are more gratitude (on average $\delta = .76$, $N_m = 45, N_f = 15$, wilcoxon-test, p-value = .02).

Thence we can conclude that in Trust Game socialization has not so much effect compared to Prisoners' Dilemma and Ultimatum Game. Here differences between sexes lead to more trust among females and more gratitude among males.

Our study is a confirmation of the fact that it is important to take into account differences between sexes in socio-economic models.

References

1. Ostrom E. Governing the commons: The evolution of institutions for collective action. Cambridge: Cambridge University Press; 1990.
2. Fehr E, Schmidt KM. A theory of fairness, competition, and cooperation. Q J Econ. 1999; 114(3): 817-868.
3. Berkman E.T., Lukinova E., Menshikov I., Myagkov M. Sociality as a Natural Mechanism of Public Goods Provision. PLoS ONE, 10(3), 2015, e0119685.
4. Menshikova O.R., Menshikov I.S., Sedush A.O. Influence of three types of socialization on the behavior of men and women in social and economic experiments. Proceedings of MIPT, 2015, pp 56-65.

On long-term average optimality in linear economic systems with unbounded time-preference rates*

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The work in devoted to the study of an average optimality problem over an infinite time horizon for linear stochastic economic systems. The agents have unbounded time-preference rates included into quadratic cost function. In both cases of positive and negative discounting we propose new optimality criteria and establish average optimal controls in the form of linear feedback laws.

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We consider a linear economic system with evolution described by a controlled stochastic process X_t defined on a complete probability space:

$$dX_t = AX_t dt + BU_t dt + G_t dw_t\,, \qquad X_0 = x, \tag{1}$$

where A, B are constant matrices; G_t is time-varying; w_t is a multidimensional Brownian motion; x is non-random; $U_t,\ t \geq 0,$ is an admissible control, i.e. an $\mathcal{F}_t = \sigma\{w_s,\ s \leq t\}$–adapted process such that there exists a solution to (1). Let us denote by $\mathcal{U}$ the set of admissible controls.

The cost functional is quadratic over the planning horizon $[0, T]$:

$$J_T^{(d)}(U) = \int_0^T f_t[(X_t' Q X_t + U_t' U_t]\,dt, \tag{2}$$

where $Q \geq 0$; f_t is a discount function, assumed to be monotone, differentiable, with $f_0 = 1$; $\phi_t = -\dot{f}_t / f_t$ defines the corresponding discount rate.

We allow the agent to have either positive or negative time-preference, i.e., $\phi_t > 0$ or $\phi_t < 0$. The impatience (or patience) in influence on her/his decisions is considered to be 'extreme' in the sense that $|\phi_t| \to \infty,\ t \to \infty$.

Examples. Weibull discount function $f_t = e^{-rt^q}$ $(q > 1,\ r > 0)$ related to highly nonlinear subjective time perception [1]. Negative double exponential discounting, when $f_t = \exp(\exp rt)$ $(r > 0)$.

Assumption $\mathcal{D}1$**.** For $\phi_t > 0$ the discount function f_t is logarithmically convex.

Assumption $\mathcal{D}2$**.** For $\phi_t < 0$ the discount rate $(-\dot{\phi}_t)/\phi_t \leq \bar{c}\phi_t$, $t \to \infty$, for some constant $\bar{c} > 0$.

First assume there exists the absolute continuous symmetric $\Pi_t \geq 0,\ t \geq 0$, which satisfies the Riccati equation

$$\dot{\Pi}_t + \Pi_t A_t + A_t'\Pi_t - \Pi_t B R^{-1} B' \Pi_t + Q = 0\,, \tag{3}$$

where $A_t := A - 1/2\phi_t \cdot I$ (I is an identity matrix).

Then we may define a feedback control law U^* by

$$U_t^* = -B'\Pi_t X_t^*\,, \tag{4}$$

where the process $X_t^*,\ t \geq 0,$ satisfies

$$dX_t^* = (A - BB'\Pi_t)X_t^* dt + G_t dw_t, \quad X_0^* = x\,. \tag{5}$$

For bounded ϕ_t the criterion based on long-run expected loss per unit of cumulative discount has been proposed in [2] to study the average optimality of U^* when $T \to \infty$. However, it would not seem to be adequate in the case considered here.

The above assumption on (3) is non-trivial. The well known sufficient conditions, e.g., control system stabilizability and detectability, all related to bounded matrices, clearly do not hold since $\|A_t\| \to \infty, \to \infty$. Moreover, A_t has specific stability properties which we describe below.

Remark. A_t is superexponentially stable if $\phi_t > 0$; superexponentially antistable if $\phi_t < 0$. The rate of stability (antistability) is ϕ_t $(-\phi_t)$.

Definition 1. Let A_t be a square matrix. Then we say that A_t is superexponentially stable with the rate $\delta_t > 0$ if $\delta_t \to \infty,\ t \to \infty$, $\|A_t\| \le \kappa\delta_t$ and $\|\Phi(t,s)\| \le \kappa_1 \exp\left(-\int\limits_s^t \delta_v\,dv\right)$, $s \le t$, where $\Phi(t,s)$ is the fundamental matrix corresponding to A_t, $\kappa, \kappa_1 > 0$ are some constants; A_t is superexponentially antistable if $-A'_t$ is superexponentially stable.

Definition 2. The pair (A_t, B_t) is said to be δ_t-superexponentially stabilizable if there exists a matrix K_t, $\|K_t\| \le \hat{c}_1\delta_t$ such that $A_t + B_tK_t$ is superexponentially stable with the rate δ_t. Similarly, the pair (A_t, C_t) is δ_t-superexponentially detectable if for F_t, $\|F_t\| \le \hat{c}_2\delta_t$, the matrix $A_t + F_tC_t$ is δ_t-superexponentially stable ($\hat{c}_1, \hat{c}_2$ are some constants).

Obviously, if A_t is δ_t-superexponentially stable then (A_t, B_t) $((A_t, C_t))$ is stabilizable (detectable) for any bounded B_t (C_t). Being valid for the case $\phi_t > 0$, it guarantees that the following statement holds true.

Theorem 1. Let Assumption $\mathcal{D}1$ hold. Then the control U^* given by (4)–(5) is a solution to

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(U)}{\int\limits_0^T (f_t/\phi_t)\|G_t\|^2\,dt} \to \inf_{U\in\mathcal{U}}.$$

Note we do not assume any bounds on G_t, hence the average optimality result remains valid even for fast-growing perturbation parameters. Because of $\mathcal{D}1$, $g_t = f_t/\phi_t$ is decreasing and may also be perceived as a discount function. Thus the denominator in the long-run average optimality criterion of Theorem 1 represents variance of cumulative extra-discounted disturbances. Due to antistability of A_t in the case of $\phi_t < 0$, we need some requirements.

Assumption 1. The pair (A_t, B_t) is $(-\phi_t)$-superexponentially stabilizable; the pair (A_t, C_t) is $(-\phi_t)$-superexponentially detectable.

Assumption 2. Let G_t and f_t be such that

$$\lim_{t\to\infty} \frac{\phi_t f_t \|G_t\|^2}{\int\limits_0^t \phi_s f_s \|G_s\|^2\, ds}\phi_t = 0\,.$$

Next we state the following result.

Theorem 2. Let Assumptions $\mathcal{D}2$, 1 and 2 hold. Then the control U^* given by (4)–(5) is a solution to

$$\limsup_{T\to\infty} \frac{EJ_T^{(d)}(U)}{\int\limits_0^T (-\phi_t) f_t \|G_t\|^2\, dt} \to \inf_{U\in\mathcal{U}}\,.$$

Again, we observe (negative) extra-discounting by $g_t = (-\phi_t)f_t > f_t$ into the average optimality criterion. Unlike the positive time-preference case, the condition relating discount rate and discounted disturbances is needed to establish the average optimality when $\phi_t < 0$. At least, we should consider only fading perturbations, i.e. $\|G_t\| \to 0$, $t \to \infty$.

References

1. Kim B.K., Zauberman G. Perception of anticipatory time in temporal discounting // Journal of Neuroscience, Psychology, and Economics. 2009. V. 2, №. 2. P. 91.–101.
2. Palamarchuk E.S. Stabilization of Linear Stochastic Systems with a Discount: Modeling and Estimation of the Long-Term Effects from the Application of Optimal Control Strategies // Mathematical Models and Computer Simulations. 2015. V. 7, №. 4. P. 381—388.

Quantile hedging of European option in multidimensional incomplete market without transaction costs (discrete time)

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Theory of European option's hedging with quantile criterion in incomplete markets without transaction costs in discrete time was considered in some articles [1–4, 6–7]. In [1] a procedure of European option's calculation with quantile criterion in one-dimensional complete market without transaction costs was offered. The procedure is based on theorem about S-representation of martingales [5]. In [4] for strictly positive contingent claim in complete one-dimensional market without transaction costs they constructed solution for the problem of quantile hedging. In [6–7] dual problems are under research: (1) direct problem is to maximaze probability of successful hedging with restriction that option's value does not exceed some given constant $x_0 > 0$; (2) dual problem is to minimaze option's value with restriction that probability of successful hedging is not less than $1 - \varepsilon$, where $\varepsilon \in (0, 1)$ is arbitrary. Unlike above stated proceedings we prove that solution of the quantile hedging problem in multidimensional incomplete market without transaction costs can be reduced to two superhedging problems.

1. Superhedging portfolio of European option. Let $\{S_t, \mathcal{F}_t\}_{t\in N_0}$ be a d-dimensional ($d < \infty$) adapted random sequence on the stochastic basis $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in N_0}, \mathsf{P}\right)$, where $N_0 \triangleq \{0, ..., N\}$, $N < \infty$ is a horizon. The sequence describes evolution of price for d risky assets. By $S_t^{(j)}$ we denote component j of d-dimensional vector S_t, $t \in N_0$. We suppose that there is one risk-free asset with zero return and initial cost 1. Let $f_N(S_\bullet)$ be a $\mathcal{F}_N$-measurable bounded random variable, $S_\bullet \triangleq (S_0, ..., S_N)$. By $\Re_N$ we denote the set of probability measures Q such that any measure $\mathsf{Q} \in \Re_N$ is equivalent to measure P. $\mathfrak{M}_N$ is the set of martingale measures. Let $\gamma_1^N \triangleq \{\gamma_t\}_{t\in N_0}$ be a d-dimensional $\mathcal{F}$-predictable sequence and $\{\beta_t\}_{t\in N_0}$ be a $\mathcal{F}$-predictable one-dimensional sequence. The sequence of pairs $\pi \triangleq (\beta_t, \gamma_t)_{t\in N_0}$ is called portfolio [5].

We denote $1_{A_N}(\omega) \triangleq \begin{cases} 1, \text{ if } \omega \in A_N \\ 0, \text{ if } \omega \notin A_N \end{cases}$, where A_N is an arbitrary $\mathcal{F}_N$-measurable set. Let us consider two calculation problems for European option with contingent claims $f_N(S_\bullet)$ and $1_{A_N}(\omega)$ in incomplete market without transaction costs [5].

Theorem 1. *Suppose* $|\Re_N \cap \mathfrak{M}_N| \geq 1$. *Than with respect to any measure* $\mathsf{Q} \in \Re_N$ *there exists solution of the calculation problem for European option with contingent claim* $f_N(S_\bullet)$ $(1_{A_N}(\omega))$.

Remark. The solution of the calculation problem for European option with contingent claim $f_N(S_\bullet)$ $(1_{A_N}(\omega))$ can be fully described as follows: $\pi^* = \{\beta_t^*, \gamma_t^*\}_{t\in N_1}$ $\left(\pi^\lambda = \left\{\beta_t^\lambda, \gamma_t^\lambda\right\}_{t\in N_1}\right)$ — self-financing portfolio, $X_t^{\pi^*}$ $\left(X_t^{\pi^\lambda}\right)$ — capital of portfolio π^* $\left(\pi^\lambda\right)$ at a moment $t \in N_0$, C_t^* $\left(C_t^\lambda\right)$ is a consumption at any moment $t \in N_1$, $X_t^{(\pi^*, C^*)} = X_t^{\pi^*} - C_t^*$

$\left(X_t^{(\pi^\lambda, C^\lambda)} = = X_t^{\pi^\lambda} - C_t^\lambda\right)$ is a capital of superhedging portfolio with consumption (π^*, C^*) $\left((\pi^\lambda, C^\lambda)\right)$ [2].

2. Quantile superhedging portfolio of European option. Let us denote: (i) $\{\chi_t, \mathcal{F}_t\}_{t\in N_0}$ – adapted random sequence with bounded variation P-a.s. [5]; (ii) $c \triangleq X_t^{\pi^*}\big|_{t=0}$.

Definition. A pair (π, χ) we call self-financing portfolio with bounded variation, where $\pi \in SF$. Capital of portfolio with bounded variation (π, χ) at a moment $t \in N_0$, denoted by $X_t^{(\pi,\chi)}$, we define by equality $X_t^{(\pi,\chi)} = X_t^\pi - \chi_t$.

Definition. By solution of the calculation problem for European option with contingent claim $f_N(S_\bullet)$ and with quantile criterion of level $1-\alpha$ (where $\alpha \in (0,1)$) in incomplete market without transaction costs with respect to any measure $\mathsf{Q} \in \Re_N$ we mean self-finansing portfolio with bounded variation $(\pi^\alpha, \chi^\alpha)$ such that it's capital $X_t^{(\pi^\alpha,\chi^\alpha)}$ at a moment N satisfies inequality $\mathsf{Q}\left(X_N^{(\pi^\alpha,\chi^\alpha)} \geq f_N(S_\bullet)\right) \geq 1-\alpha$. Portfolio $(\pi^\alpha, \chi^\alpha)$ we will name quantile superhedging portfolio of level $1-\alpha$.

Theorem 2. *Suppose $f_N(S_\bullet)$ is a $\mathcal{F}_N$-measurable bounded random variable and $|\Re_N \cap \mathfrak{M}_N| \geq 1$. Suppose also that for any $\alpha \in (0,1)$ there are $\lambda_t^{(j)}(\alpha) \in \mathbb{R}^+$, $j = \overline{1,d}$, $t \in N_0$ such that with respect to any measure $\mathsf{Q} \in \Re_N$ the following inequality is true*

$$\mathsf{Q}\left(\bigcap_{t=1}^{N}\bigcap_{j=1}^{d}\left\{S_t^{(j)} \geq \lambda_t^{(j)}(\alpha)\right\}\right) \geq 1-\alpha.$$

Then there exists solution of the calculation problem for European option with quantile criterion of level $1-\alpha$.

Remark. Quantile superhedging portfolio of level $1-\alpha$, i.e. $(\pi^\alpha, \chi^\alpha)$, has the form: $\gamma_t^\alpha = \gamma_t^* - c\gamma_t^\lambda$, $\beta_t^\alpha = \beta_t^* - c\beta_t^\lambda$, $\chi_t^\alpha = C_t^* - cC_t^\lambda$. It's initial capital $X_0^{(\pi^\alpha,\chi^\alpha)} = c\left(1 - X_0^{\pi^\lambda}\right)$.

3. Minimax quantile hedging portfolio of European option. In presentation the solution of European calculation problem with respect to the "worst-case" measure $\mathsf{Q}^* \notin \Re_N$ will be given (see. [2]). It is proved that with respect to Q^* initial incomplete market is complete and Q^* is discreet. This facts allowed us to construct new examples of European option's calculation with quantile criterion in incomplete market with respect to Q^*.

References

1. Zverev O.V. Calculation of European option in complete (B, S)-market with quantile criterion [in Russian] // Proceedings of scientific and technical conference for students, postgraduate students and specialists of MSIEM. Moscow, 2007. P. 31.
2. Zverev O.V., Khametov V.M. Minimax hedging of European options in incomplete markets (discreet time) [in Russian] // Review of applied and industrial mathematics. 2011. V. 18, No. 1. P 26–54.
3. Zverev O.V., Khametov V.M. Minimax hedging of European options in incomplete markets (discreet time) [in Russian] // Review of applied and industrial mathematics. 2011. V. 18, No 2. P 193–204.
4. Melniko A.V., Volkov S.N., Nechaev M.M. Mathematics of financial obligations [in Russian]. Moscow: HSE, 2001.
5. Shiryaev A.N. Fundamentals of Stochastic Financial Mathematics. Volume 2: Theory [in Russian]. Moscow: Moscow, 1998.
6. Föllmer H., Schied A. Stochastic Finance. An Introduction in Discrete Time. Berlin: Walter de Gruyter, 2004.
7. Föllmer H., Leukert P. Quantile hedging // Finance and Stochastics. 1999. Vol. 3. P. 251–273.

OR in finance and banking

Solution of two-parameter consumption-investment problem

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This work concerns the consumption-investment problem with stochastic interest rate r_t [1] and volatility η_t [2, 3]. The resulting model is close to [4]. More detailed description see in [5].

We consider a model with parameters η_t и r_t, whose dynamics are driven by Cox-Ingersoll-Ross (CIR) model [6]:

$$\begin{aligned} d\eta_t &= l(N-\eta_t)dt + \sigma_\eta \sqrt{\eta_t} dZ_1(t), \\ dr_t &= k(R-r_t)dt + \sigma_r \sqrt{r_t} dZ_2(t), \end{aligned}$$

where σ_η, σ_r, l, N, k, R are positive constants. $Z_1(t)$ и $Z_2(t)$ are independent standard Wiener processes. Furthermore it is assumed that $2lN > \sigma_\eta^2$ and $2kR > \sigma_r^2$.

Financial market consists of three assets, which are traded continuously over $[0,T]$. One is a risk-free asset with interest rate r_t and other two are risky assets, whose price processes S_{1t}, S_{2t} satisfy equations

$$\begin{aligned} \frac{dS_{1t}}{S_{1t}} &= (r_t + m\eta_t)dt + \sigma_1 \sqrt{\eta_t} dZ_1(t), \\ \frac{dS_{2t}}{S_{2t}} &= (r_t + nr_t)dt + \sigma_2 \sqrt{r_t} dZ_2(t), \end{aligned}$$

where m, n, σ_1, σ_2 are positive constants.

Assume that the investor has a power utility function $u(C) = C^\gamma/\gamma$. At time t he invests in risky assets and consumes fractions $\pi_{1,2t}$ and c_t respectively.

Mathematically, the investor wishes to maximize the following expected utility:

$$U(W,\eta,r) = \max_{(c_s \geq 0,\ \pi_{1,2\ s})|_{s=0}^{T}} E_0 \left[\alpha \int_0^T e^{-\delta s} \frac{(c_s W_s)^\gamma}{\gamma} ds + (1-\alpha) e^{-\delta T} \frac{W_T^\gamma}{\gamma} \right],$$

where $e^{-\delta t}$ is a discount coefficient.

Using the dynamic programming principle, one can get the Hamilton-Jacobi-Bellman equation

$$H(W,\eta,r,t) = \max_{(c_s \geq 0,\ \pi_{1,2\ s})|_{s=t}^{T}} E_t \left[\alpha \int_t^T e^{-\delta s} \frac{(c_s W_s)^\gamma}{\gamma} ds + (1-\alpha) e^{-\delta T} \frac{W_T^\gamma}{\gamma} \right].$$

Let us introduce the following notation

$$\begin{aligned}
D_f &= \frac{\sigma_1^2 l^2 - \gamma\left(\sigma_1 l + \sigma_\eta m\right)^2}{(1-\gamma)\sigma_1^2},\ D_g = \frac{\sigma_2^2 k^2 - \gamma\left[\left(\sigma_2 k + \sigma_r n\right)^2 + 2\sigma_2^2\sigma_r^2\right]}{(1-\gamma)\sigma_2^2}, \\
\lambda_{1,2} &= \frac{1}{\sigma_\eta^2}\left(l - \frac{\gamma}{1-\gamma}\frac{\sigma_\eta}{\sigma_1} m\right) \pm \frac{\sqrt{D_f}}{\sigma_\eta^2},\ X = \frac{\sigma_\eta^2(\lambda_1 - \lambda_2}{2}, \\
\lambda_{3,4} &= \frac{1}{\sigma_r^2}\left(k - \frac{\gamma}{1-\gamma}\frac{\sigma_r}{\sigma_2} n\right) \pm \frac{\sqrt{D_g}}{\sigma_r^2},\ Y = \frac{\sigma_r^2(\lambda_3 - \lambda_4)}{2}.
\end{aligned}$$

Then the Hamilton-Jacobi-Bellman equation has a solution of the form

$$H(W,\eta,r,t) = e^{-\delta t}\frac{W^\gamma}{\gamma} F^{1-\gamma}(\eta,r,t),$$

where

$$F(\eta,r,t) = \alpha^{1/(1-\gamma)} \int_t^T G(\eta,r,s)ds + (1-\alpha)^{1/(1-\gamma)} G(\eta,r,t),$$

$$G(\eta,r,t) = e^{f(t)\eta + g(t)r + h(t)},$$

and functions $f(t)$, $g(t)$ and $h(t)$ are defined as

$$f(t) = \frac{\lambda_1\lambda_2\left(e^{X(T-t)}-1\right)}{\lambda_1 e^{X(T-t)}-\lambda_2},\ g(t) = \frac{\lambda_3\lambda_4\left(e^{Y(T-t)}-1\right)}{\lambda_3 e^{Y(T-t)}-\lambda_4},$$

$$h(t) = lN\int\limits_t^T f(s)ds + kR\int\limits_t^T g(s)ds - \frac{\delta}{1-\gamma}(T-t).$$

Optimal investor strategies are equal to

$$\begin{aligned}
\pi_1^*(\eta,r,t) &= \frac{m}{(1-\gamma)\sigma_1^2} + \frac{\sigma_\eta}{\sigma_1}\frac{F'_\eta(\eta,r,t)}{F(\eta,r,t)},\\
\pi_2^*(\eta,r,t) &= \frac{n}{(1-\gamma)\sigma_2^2} + \frac{\sigma_r}{\sigma_2}\frac{F'_r(\eta,r,t)}{F(\eta,r,t)},\\
c^*(\eta,r,t) &= \frac{\alpha^{1/(1-\gamma)}}{F(\eta,r,t)}.
\end{aligned}$$

Furthermore, assuming a number of restrictions one can estimate the expected utility for infinite time horizon as

$$\widehat{U}(W,\eta,r) = \frac{W^\gamma}{\gamma}\left(\frac{K}{L}\right)^{1-\gamma} e^{(1-\gamma)(\lambda_2\eta+\lambda_4 r)}$$

and the optimal strategies are equal to

$$\widehat{\pi_1}^* = \frac{m}{(1-\gamma)\sigma_1^2} + \frac{\sigma_\eta}{\sigma_1}\lambda_2;\ \widehat{\pi_2}^* = \frac{n}{(1-\gamma)\sigma_2^2} + \frac{\sigma_r}{\sigma_2}\lambda_4;\ \widehat{c}^* = \frac{L}{K}e^{-\lambda_2\eta-\lambda_4 r},$$

where

$$K = \left(\frac{\lambda_1-\lambda_2}{\lambda_1}\right)^{2lN/\sigma_\eta^2}\left(\frac{\lambda_3-\lambda_4}{\lambda_3}\right)^{2kR/\sigma_r^2},\ L = \frac{\delta}{1-\gamma} - lN\lambda_2 - kR\lambda_4.$$

References

1. Korn R., Kraft H. A stochastic control approach to portfolio problems with stochastic interest rates// SIAM Journal of Control and Optimization, 2001. V. 40. P. 1250–1269
2. Heston S.L. A closed-form solution for options with stochastic volatility with applications to bonds and currency options// The Review of Financial Studies. 1993. V. 6. № 2. P. 327–343.

3. Fleming W.H., Pang T. An Application of Stochastic Control Theory to Financial Economics// SIAM Journal of Control and Optimization. 2004. V. 43. № 2. P. 502–531.
4. Chang H., Rong X. An investment and consumption problem with CIR interest rate and stochastic volatility// Abstract and Applied Analysis. 2013. Special Issue(2012). Article ID 219397. P. 1–12.
5. Liu J. Portfolio selection in stochastic environments// The Review of Financial Studies. 2007. V. 20. № 1. P. 1–39.
6. Cox J.C., Ingersoll J.E.Jr., and Ross S.A. An Intertemporal General Equilibrium Model of Asset Prices// Econometrica. 1985. V. 53. № 2. P. 363–384.

Futures position management based on multistage stochastic programming

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This report introduces futures portfolio management models. These models take into account an initial margin for futures. They consider a long-term investment horizon which can be transfered in the future in a case of low probability to achieve the required portfolio value at the end of investment horizon. Analyzed models allow trading in futures of different expirations. Buy/sell commission is deducted from the account for each trade. Variation margin is calculated each trading day. Thus, such portfolio management models are close to the real market conditions.

This work presents results of experiments, where the portfolio includes futures of different expiration dates on a single underlying asset. We consider three underlying assets: RTS index, Gazprom and Sberbank. The prices of the relevant futures have been taken from Moscow exchange website [1]. The price of the underlying asset is modeled using ARIMA-GJR model. It is a GARCH model with a leverage effect which stems from the fact that losses have a greater influence on future volatilities than gains.

$$\sigma_t^2 = K + \delta\sigma_{t-1}^2 + \alpha\epsilon_{t-1}^2 + \phi\epsilon_{t-1}^2 I_{t-1} \tag{1}$$

where $I_{t-1} = 0$ if $\epsilon_{t-1} \geq 0$, and $I_{t-1} = 1$ if $\epsilon_{t-1} < 0$.

The problem of portfolio optimization is formulated as a problem of multistage stochastic programming [2], [3]. Rebuilding the portfolio in accordance with the solution of the optimization problem is done every trading day. For this optimization in a particular trading day a tree of scenarios of possible price movements of the underlying asset with the corresponding probabilities of the scenarios is built using ARIMA-GJR model. Next, the problem of dynamic portfolio optimization is solved using this tree. The results of the optimization are recommendations to buy and sell contracts in the root node of the tree, which minimize the risk of failure to achieve the required value of the portfolio by a certain date.

The calculations in each simulated trading day on the futures market include: commission for the transactions, calculation of variation margin, monitoring the probability of reaching the required portfolio value.

Let u be a desired value of the portfolio at the terminal moment of time;

g_ν is a value determined for each scenario ν based on the following inequalities:

$$g_\nu + W_{T_\nu} \geq u, g_\nu \geq 0 \tag{2}$$

Then the optimization criterion can be written as follows:

$$min \sum_{\nu=1}^{N} g_\nu p_\nu, \tag{3}$$

p_ν is the probability of the scenario ν.

This criterion presents a minimum of the expectation value of g_ν. So, solving the optimization problem the portfolio with minimal expected possible gap is constructed.

The result of modelling 2 months trading for portfolio which included futures on Sberbank is presented in Fig. 1. The required capital was 120000 roubles, the initial capital was 100000 roubles,the commission per trade was 2 roubles, maintenance margin was 1400 roubles for a contract. For a terminal moment of time the value of the portfolio was 117334. It is less than the required value but still the portfolio showed a profit.

On the whole, we simulated 1-year traces of portfolio management for RTS index futures, futures on Gazprom and Sberbank with different maturities.

References

1. `http://moex.com/`.

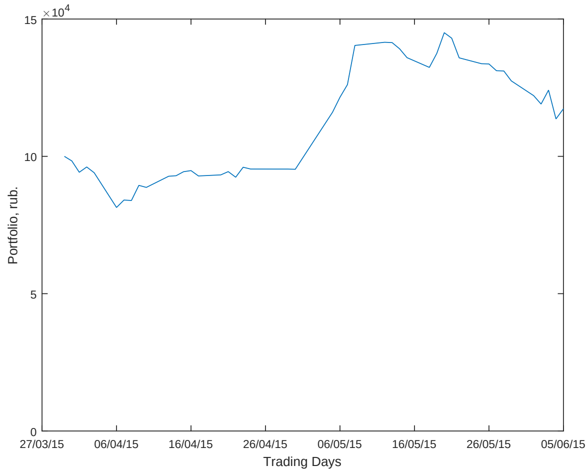

Fig. 1. Result for portfolio of futures on Sberbank.

2. Shapiro A., Dentcheva D. Ruszczynski A. Lectures on Stochastic Programming: modeling and theory// MPS-SIAM Series on Optimization. - 2009.
3. Golembiovsky, D. and Abramov, A., Option portfolio management as a chance constrained problem. In Stochastic Programming: Applications in Finance, Energy, Planning and Logistics, edited by H. Gassmann, S. W. Wallace, W. T. Ziemba, 2013, 155-172 (World Scientific).

Maximum likelihood estimator for default rate of the credit portfolio

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Banks must calculate reserves for possible credit portfolio losses in accordance with Basel II requirements [3] by the following formula (1):

$$\text{Reserves} = EAD * PD * LGD, \tag{1}$$

where EAD – the Exposure at Default, PD – Default Probability of credit; LGD (Loss Given at Default) -– non-payment of funds by credit when default occurs.

Banks usually uses different delinquency indexes for control of default risk level.

There is offered to use the maximum likelihood estimator for samples from the stratified set [1,2] to estimate the credit portfolio default rate.

Let $t_0 < t_1 < \cdots < t_i < t_N$ are the given calendar date, here the month's last days are considered. Let $V_i(t)$ is a vintage (= set of loans, opened during time period $[t_{i-1}, t_i]$) at the current moment t, and V_i is the vintage $V_i(t)$ at the moment $t = t_i, \quad i = 1, \ldots, N$. It is clear that $V_i(t) \cap V_j(t) = \emptyset, i \neq j, V_i(t) = \emptyset$, if $t < t_i, i = 1, \ldots, N$.
$\cup_{i=1}^N V_i(t)$ is a credit portfolio at moment t.

For vintage $V_i = V_{i,D} \cup V_{i,ND}$, where $V_{i,D}(V_{i,ND})$ is the set of defaulted (non-defaulted) credits in the vintage. Quantity $K(V_{i,D})$ of defaulted credits and quantity $K(V_{i,ND})$ of non-defaulted credits in vintage V_i are unknown, but vintage size $K(V_i) = K(V_{i,D}) + K(V_{i,ND})$ is known. Let β_{it} is the rate of observed defaults in $V_i(t)$ at the moment $t, i = 1, \ldots, N$.

Maximum likelihood estimator $\widehat{\beta}_{it}$ (from [1,2]) might be used for assessing default rate of a credit portfolio $\bigcup_{i=1}^N V_i(t)$ at the moment t.

It is offered the following maximum likelihood estimator of default probability PD_t for the given moment t:

$$\widehat{PD}_t = \left(\sum_{i=1}^N \widehat{\beta_{it}} K\left(V_i(t)\right) \right) / \sum_{i=1}^N K\left(V_i(t)\right).$$

References

1. G. I. Ivchenko and S. A. Khonov An asymptotic estimate for stratified finite populations. Diskr. Mat., 1989, Volume 1, Issue 3, Pages 87—95.
2. G. I. Ivchenko and S. A. Khonov Statistical estimation of the composition of a finite set. Diskr. Mat., 1996, Volume 8, Issue 1, Pages 3—40.
3. Basel II: International Convergence of Capital Measurement and Capital Standards (2006), p.86.

Bounds on the value of American option on difference of two assets*

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An American call option on the difference of two assets (a two side Margrabe option [1]) provides its holder the right to exchange one asset for another at any time prior to expiration T at strike $K_i,\ i = 1,2$ depending on the asset. An upper bound is constructed using a method based on the integral formula of option value [2]. A lower bound is derived by Monte Carlo simulations using exercise boundary approximation as a decision rule.

We consider the asset values $S_i(t),\ i = 1,2$ satisfy the equations of geometrical Brownian motion $dS_i(t) = S_i(t)(\alpha_i dt + \sigma_i dz_i(t)),\ i = 1,2,$ where $z_i(t),\ i = 1,2$ are standard Wiener processes ($z_i(0) = 0$) with constant correlation $|\rho| < 1$, $r > 0$ is a bank interest rate, $\alpha_i = r - \delta_i$ are the average rates of return, σ_i^2 are the average volatilities, $\delta_i > 0$ are the dividends paid on the ith asset. The payoff at time t is given by $f(S(t)) = \max\limits_{i=1,2}(S_i(t) - S_{3-i}(t) - K_i)_+$ where $a_+ = \max(a,0)$ and $S(t) = (S_1(t), S_2(t))$.

Let $S = (S_1, S_2)$. The initial option value $F(S,t)$ can be determined as an upper bound of mean discounted payoffs over all the exercise decision rules: $F(S,t) = \sup_{\tau\in[t,T]} E[e^{-r(\tau-t)} f(S(\tau))|S(t) = S]$.

The optimal decision rule is given by [3]

$$\tau^* = \min\left(t \mid F(S_1(t), S_2(t), t) = f(S_1(t), S_2(t), t), T\right)$$

and defines the immediate exercise region

$$\mathcal{E}(t) = \left\{S \in \mathbb{R}_+^2 \mid F(S,t) = f(S,t),\ \max(S_1(t), S_2(t)) > 0\right\}.$$

It is shown that the immediate exercise region $\mathcal{E}$ consists of two disjoint subregions:

$$\mathcal{E}_i(t) = \left\{S \in \mathcal{E}(t) \middle|\ S_i(t) - S_{3-i}(t) > \frac{K_i - K_{3-i}}{2}\right\},\ i = 1,2.$$

Let $G_i(S_{3-i}, t)$ denote the border of the subregion $\mathcal{E}_i(t),\ i = 1,2$. It is shown that $G_i(S_{3-i}, t)$ are convex nondecreasing functions and the graph

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of $G_i(S_{3-i}, t)$ approaches asymptotically the line $S_i = c_i(t)S_{3-i} + w_i(t)$, where $c_i(t) \geqslant 1$ and $w_1(t) \geqslant (K_1 - K_2)/2$, $w_2(t) \geqslant -c_2(t)(K_1 - K_2)/2$ in case of $K_1 > K_2$.

To derive the coefficients $c_i(t)$, $w_i(t)$ and derivatives $G_i'(0,t)$ the integral formula of option value is used [4]:

$$F(S,t) = C(S,t) + \sum_{i=1}^{2} \int_0^T e^{-rt} \int_{M_i(t)} \Big(\delta_i S_i e^{\tilde{\alpha}_i t + \sigma_i \sqrt{t} x_i} -$$

$$-\delta_{3-i} S_{3-i} e^{\tilde{\alpha}_{3-i} t + \sigma_{3-i}\sqrt{t} x_{3-i}} - rK_i\Big) \psi(x) dx dt, \; i = 1, 2,$$

$$M_i(t) = \left\{ x \in \mathbb{R}^2_+ \mid S_i e^{\tilde{\alpha}_i t + \sigma_i \sqrt{t} x_i} \geqslant G_i(S_{3-i} e^{\tilde{\alpha}_{3-i} t + \sigma_{3-i}\sqrt{t} x_{3-i}}, t) \right\}, \quad (1)$$

where $C(S,t) = e^{-r(T-t)}\mathrm{E}[f(S(T))]$ is a price of corresponding European option, $x = (x_1, x_2)$, $\psi(x)$ is a bivariate normal density function. Let

$$\tilde{\alpha}_i = \alpha_i - \frac{\sigma_i^2}{2}, \; \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2, \; \tilde{\alpha} = \alpha_1 - \alpha_2, \; \hat{\alpha} = \tilde{\alpha}_1 - \tilde{\alpha}_2,$$

$$d_i(S_i) = \frac{\ln(S_i/K_i) + (\tilde{\alpha}_i + \sigma_i^2)T}{\sigma_i\sqrt{T}}, \; \tilde{d}_i(S_i) = \frac{\ln(S_i/K_i) + (\tilde{\alpha}_i + \rho\sigma_1\sigma_2)T}{\sigma_i\sqrt{T}},$$

$$a = \frac{\rho}{\sqrt{1-\rho^2}}, \; r_i = r - \tilde{\alpha}_i, \; \zeta_i = \frac{1}{\sigma_i\sqrt{1-\rho^2}}, \; b_i = \tilde{\alpha}_i\zeta_i, \; b_i' = (\tilde{\alpha}_i + \sigma_i^2)\zeta_i,$$

$$\delta_{i,3-i} = \delta_i - \tilde{\alpha}_{3-i} - \rho\sigma_1\sigma_2, \; d(c_i) = \frac{\ln(c_i) + ((-1)^{3-i}\tilde{\alpha} + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

$$\tilde{d}(c_i) = \frac{\ln(c_i) + ((-1)^{3-i}\tilde{\alpha} - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \; \hat{d}(c_i) = \frac{\ln(c_i) + (-1)^{3-i}\hat{\alpha}T}{\sigma\sqrt{T}},$$

$$b_i'' = (\tilde{\alpha}_i + \rho\sigma_1\sigma_2)\zeta_i, \; \Lambda'_{i,3-i} = I(a, b_i', \sigma_{3-i}, 0, \delta_{i,3-i}) - I(a, b_i', 0, 0, \delta_i),$$

$$\lambda_i = \frac{rK_i}{G_i(0,0)}, \; \Lambda_{i,3-i} = I(a, b_i, \sigma_{3-i}, 0, r_{3-i}) - I(a, b_i, 0, 0, r),$$

$$I(a,b,c,d,\delta) = \frac{e^{-\frac{d(b+ac)}{a^2+1}}}{\sqrt{\eta}} \left[e^{-\frac{|d|\sqrt{\eta}}{a^2+1}} \Phi\left(\sqrt{\frac{\eta T}{a^2+1}} - \frac{|d|}{\sqrt{(a^2+1)T}}\right) - \right.$$

$$\left. -e^{\frac{|d|\sqrt{\eta}}{a^2+1}} \Phi\left(-\sqrt{\frac{\eta T}{a^2+1}} - \frac{|d|}{\sqrt{(a^2+1)T}}\right)\right],$$

$$\eta = (b+ac)^2 + (-c^2 + 2\delta)(a^2+1) > 0.$$

$$J(a,b,d,\delta)=h(d)-e^{-\delta T}\Phi\left(\frac{b\sqrt{T}+\frac{d}{\sqrt{T}}}{\sqrt{a^2+1}}\right)+$$
$$+\frac{1}{2}\left(\frac{b}{\sqrt{\eta(0)}}-2h(d)+1\right)e^{-\frac{db+d\sqrt{\eta(0)}}{a^2+1}}\Phi\left(\sqrt{\frac{\eta(0)T}{a^2+1}}-\frac{d}{\sqrt{(a^2+1)T}}\right)-$$
$$-\frac{1}{2}\left(\frac{b}{\sqrt{\eta(0)}}+2h(d)-1\right)e^{-\frac{db-d\sqrt{\eta(0)}}{a^2+1}}\Phi\left(-\sqrt{\frac{\eta(0)T}{a^2+1}}-\frac{d}{\sqrt{(a^2+1)T}}\right),$$

where $\eta(0)=b^2+2\delta(a^2+1),\ \delta>0,\ d\geqslant 0,\ h(0)=1/2,\ h(d)=1$ if $d>0$.

It is shown that $G_i'(0,0)$ is equal to

$$\frac{1-J(a,b_i'',0,\delta_{3-i})-e^{-\delta_{3-i}T}\Phi(\tilde{d}_i(G_i(0,0)))}{1-J(a,b_i,0,\delta)+\delta_i\zeta_i\Lambda'_{i,3-i}-\lambda_i\zeta_i\Lambda_{i,3-i}-e^{-\delta_iT}\Phi(d_i(G_i(0,0)))}.$$

The coefficients $c_i(t)$ and $w_i(t),\ i=1,2$ for any $0<t<T$ are derived as a solution of the system of nonlinear equations.

Note that function $\underline{G}_i(S_{3-i},t) = \max[\underline{G}_i'(0,t)S_{3-i} + \underline{G}_i(0,t), c_i(t)S_{3-i}+w_i(t), (\delta_{3-i}S_{3-i}+rK_i)/\delta_i]$ is not greater than $G_i(S_{3-i},t),\ i=1,2$. Let $\bar{M}_i$ be M_i substituting $G_i(S_{3-i},t)$ for $\underline{G}_i(S_{3-i},t)$. Then M_i is contained in $\bar{M}_i$. An upper bound of the option value can be derived by substituting M_i for $\bar{M}_i$ into (1).

For example, let $r=0.05;\ \delta_1=\delta_2=0.01;\ \sigma_1=0.2;\ \sigma_2=0.1;\ \rho=0.5;\ K_1=8;\ K_2=5;\ S_1=15; S_2=5;\ T=3$ then: $c_1(0)=c_2(0)=1.775;\ w_1(0)=13.97;\ w_2(0)=5.39$.

The lower bound of the option is calculated using the exercise rule $\tau^0=\min[\min\{t|S(t)\in\bar{M}_1(t)\bigcup\bar{M}_2(t)\},T]$ and Monte-Carlo simulation. An upper bound is equal to 3.428, and a lower bound is equal to 3.424.

References

1. Margrabe W. The value to exchange one asset for another // Journal of Finance. 1978. V. 33, № 1. P. 177–186.
2. Vasin A.A., Morozov V.V. Investment decision under uncertainty and evaluation of American options // International Journal of Mathematics, Game Theory and Algebra. 2006. V. 15, № 3. P. 323–336.
3. Shiryaev A.N. Optimal Stopping Rules. New-York: Springer-Verlag, 1978.

4. Broadie M., Detemple J. The valuation of American options on multiple assets // Mathematical Finance. 1997. V. 7, № 3. P. 241–285.

On VaR-type risk measures under hedging of American contingent claims*

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In this study we research game problems between seller and buyer of an American contingent claim, discuss properties and optimization complexity of value-at-risk measure and expected shortfall and develop decomposition methods to solve these problems much more faster.

We consider a multiperiod model of the financial market which leads to a large scale nature of the given problems because a number of buyer's strategies grows overexponentially. Therefore, decomposition of these games turns out to be our fundamental goal. As for the main optimization problem, we look for the optimal investment strategy which produces the minimal losses associated with imperfect (or incomplete) hedging of American contingent claim. It consists in finding a minimax value of a specific zero-sum game.

We suppose that security trading in financial market occurs in deterministic moments of time and a market has a finite number of scenarios (however, it may be quite huge). There are no transaction costs during the trades. The market consists of a few tradable securities with known probability distribution of prices. One security is riskless (a bank deposit or a bond), it has strictly positive prices. The number of risky securities (stocks) can be any.

The set of states $\mathcal{N}$ of the market has a tree structure. It is divided into pairwise disjoint subsets of states $\mathcal{N}_t$ which may occur at specific time moments $t = 0, ..., T$. The set $\mathcal{N}_0$ contains the only element – a root of the tree denoted by 0. Every node $n \in \mathcal{N}_t$, where $t = 1, ..., T$, has a unique parent node.

We state a zero-sum game between two players: a seller of the contingent claim and its buyer. The seller is an investor in wide sense, he builds a trading strategy to hedge the American contingent claim. The buyer

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exercises the claim in some moment of time (i.e. obliges the seller to pay the claim value using his right specified in a contract).

The main feature of an American contingent claim is an uncertain moment of exercise. So, American claims may be exercised by its buyer at any time $t = \{0, ..., T\}$ up to expiration date. Exercise time is usually considered as an uncertain factor in investment problems. Besides, it means stopping time for random processes of the claim and the losses. Next, we define strategies of players.

Investor strategy is a self-financing portfolio process, i.e. he does not spend money and does not get any revenue from outside. Portfolio value process $V = \{V(t)\}$ corresponds to a trading strategy. A random variable $V(t)$ takes values V_n equal to scalar products of price and portfolio vectors. We suppose that there are no arbitrage opportunities in the market, i.e. there are no trading strategies, such that the investor loses nothing and yields a positive profit with a positive probability. We consider only admissible trading strategies, the ones which prevent the investor from ruin.

Buyer's strategy is a moment of time when the contingent claim is exercised. Let us describe it with a random variable τ. For each sequence of consecutive states $(n_0, ..., n_T)$ it produces the only state, where stopping occurs. Let $\mathcal{N}_\tau$ be a set of these states. We show that a set of buyer's strategies grows overexponentially while a number of trading periods T increases.

An American contingent claim is described with a non-negative stochastic process $F = \{F(t)\}$. The examples of a contingent claim are payments on option, forward or futures contracts. Portfolio strategy hedges an American contingent claim F exercised in time τ if the portfolio values $V_n \geq F_n$ for all $n \in \mathcal{N}_\tau$. Perfect hedging (with probability one) of an American contingent claim generally requires considerable initial endowment from the seller.

Suppose that the seller does not have a necessary sum for perfect hedging and decides to manage with less initial endowment taking the risk of future losses. So, if the claim is exercised in state $n \in \mathcal{N}$ of the market, then seller's losses are equal to $(F_n - V_n)^+ = \max\{F_n - V_n;\ 0\}$.

In the first part of this research we propose value-at-risk (VaR) as a risk measure to estimate the losses from imperfect hedging. It is equal to the minimum value such that the expected losses do not exceed it with a specified probability. In other words, VaR corresponds to the amount of uninsured risk which the seller can take; see [2]. This measure is recommended primarily for monitoring market risks and effectiveness of hedging strategies. VaR approach of risk estimation was also widely studied in [3]. We evaluate seller's losses in exercise time τ using the value-at-risk function:

$$\mathrm{VaR}_\alpha\big((F(\tau) - V(\tau))^+\big) = \min\{B \in \mathbb{R} \,|\, \mathbb{P}\big((F(\tau) - V(\tau))^+ \leq B\big) \geq \alpha\},$$

where α is a preset level of significance.

We state the optimization problem from the seller's side to find an optimal investment strategy V which imperfectly hedges contingent claim F and minimizes a loss function VaR_α under uncertain exercise time τ. The given problem consists in finding a minimax value of the game and can be formulated in the following way:

$$\min_{V} \max_{\tau \in \mathcal{T}} \mathrm{VaR}_\alpha((F(\tau) - V(\tau))^+)$$
$$V_n \geq 0, \ \forall\, n \in \mathcal{N}.$$

We incorporate binary variables x which characterize the decisions coupled with probability constraints in a definition of VaR and formulate the original problem as a mixed-integer programming problem. Then, we prove the existence of optimal trading strategy such that x^* has a monotonic nature over time. Namely, we show that

$$x^*(t) \geq x^*(t+1), \ \forall\, t = 0, ..., T-1.$$

Then, we analyze the similar optimization problem using expected shortfall as a risk measure (see [1]) and discuss this problem from the buyer's perspective. It allows us to take into consideration not only the fact of losses but the amount of them as well. Here the problem consists in finding maximin value. We show that considered utility functions usually but not always have saddle points.

The obtained results allow to substantially decrease a number of constraints in the original problem and let us turn to an equivalent mixed integer problem with admissible dimension. Thus, we exclude the uncertainty associated with the time of exercising the contingent claim.

The outcomes of this study can be useful for software systems development in financial institutions which deal with valuation and hedging of contingent claims, building trading strategies. Consideration of discrete models of a financial market for dealing with investment problems allowed to apply methods of mathematical programming and game theory.

References

1. Acerbi C. and D. Tasche. Expected Shortfall: A Natural Coherent Alternative to Value at Risk // Economic Notes. 2002. V. 31, N 2. P. 379-388.
2. Rockafellar R.T. and S. Uryasev. Optimization of conditional value-at-risk // Journal of Risk. 2000. V. 2, N 3. P. 21–41.
3. Rockafellar R.T. and S. Uryasev. Conditional value-at-risk for general loss distributions // Journal of Banking & Finance. 2002. V. 26. P. 1443–1471.

OR in insurance and risk-management

Impact of risky investments on the solvency of insurers in a model with stochastic premiums

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We consider the mathematical insurance model with stochastic premiums and risky investments; for its detailed and complete investigation, see [1–3] and references therein.

1. For the modified Cramér-Lundberg model with stochastic premiums, the continuous-time risk process has the form

$$R_t = u + \sum_{i=1}^{N_1(t)} C_i - \sum_{j=1}^{N(t)} Z_j, \qquad t \geq 0. \tag{1}$$

Here, R_t is the surplus of an insurance company at time t; u is the initial surplus (IS); the first sum on the right-hand side represents the aggregate premiums up to time t; $N_1(t)$ is a homogeneous Poisson process (HPP) with intensity $\lambda_1 > 0$ ($\mathbf{E}N_1(t) = \lambda_1 t$, $N_1(0) = 0$) that, for any $t > 0$, determines the number of premiums charged over the time interval $(0, t]$; C_1, C_2,... are independent identically distributed (IID) random variables with a distribution function $G(y)$ ($G(0) = 0$, $\mathbf{E}C_1 = n < \infty$) that determine the premium sizes and are assumed to be independent of

$N_1(t)$; and the second sum is the aggregate claims; $N(t)$ is a HPP with intensity $\lambda > 0$ ($\mathbf{E}N(t) = \lambda t$, $N(0) = 0$) that, for any $t > 0$, determines the number of claims over the time interval $(0, t]$; Z_1, Z_2,... are IID random variables with a distribution function $F(x)$ ($F(0) = 0$, $\mathbf{E}Z_1 = m < \infty$) that determine the claim sizes and are independent of $N(t)$. The aggregate premium and aggregate claim processes are also assumed to be independent.

Let now the surplus be invested continuously in stocks with prices described by the stochastic differential equation (SDE) $dS_t = S_t(a\,dt + b\,dw_t)$, $t \geq 0$. Here, S_t is the stock price at time t, $0 < a$ is the expected stock return rate, $0 < b$ is the volatility parameter, and $\{w_t\}$ is a standard Wiener process, or a Brownian motion.

Then the dynamics of the surplus (resulting risk process) is described by the initial value problem for an SDE:

$$dX_t = X_t\,(a\,dt + b\,dw_t) + dR_t, \quad t \geq 0, \quad X_0 = u. \tag{2}$$

Here, X_t is the portfolio value at time t and R_t is the risk process (1).

As a measure of the solvency of an insurance company, we use the survival probability (SP) $\varphi(u)$ (as a function u) in infinite time: $\varphi(u) = \mathbf{P}\{X_t \geq 0,\, t > 0\}$, where $X_0 = u$ for $u \geq 0$; for $u < 0$, we set $\varphi(u) \equiv 0$.

The equation for $\varphi(u)$ of the resulting risk process (2) has the form:

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) = \lambda\left[\varphi(u) - \textstyle\int_0^u \varphi(u-x)dF(x)\right] +$$

$$+\lambda_1\left[\varphi(u) - \textstyle\int_0^\infty \varphi(u+y)dG(y)\right], \quad u \in \mathbb{R}_+.$$

2. Assuming that the premium and claim sizes have exponential distributions, $F(x) = 1 - \exp(-x/m)$, $G(y) = 1 - \exp(-y/n)$, $m, n > 0$, we formulate the constrained singular nonlocal problem (see [1,3]):

$$\begin{gathered}(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] - \\ -\lambda_1[\varphi(u) - (I_n\varphi)(u)] = 0, \quad u > 0,\end{gathered} \tag{3}$$

$$|\lim_{u\to+0}\varphi(u)| < \infty, \qquad \lim_{u\to+0}[u\varphi'(u)] = 0, \tag{4}$$

$$(\lambda + \lambda_1)\lim_{u\to+0}\varphi(u) = \lambda_1(I_n\varphi)(0), \tag{5}$$

$$0 \leq \varphi(u) \leq 1 \qquad \forall u \in \mathbb{R}_+, \tag{6}$$

$$\lim_{u\to+\infty} \varphi(u) = 1, \qquad \lim_{u\to+\infty} \varphi'(u) = 0. \tag{7}$$

Here, J_m and I_n are Volterra and non-Volterra integral operators, respectively,

$$(J_m\varphi)(u) = \frac{1}{m}\int_0^u \varphi(u-x)\exp(-x/m)dx,$$

$$(I_n\varphi)(u) = \frac{1}{n}\int_0^\infty \varphi(u+y)\exp(-y/n)dy,$$

where J_m, I_n: $C[0,\infty) \to C[0,\infty)$ and $C[0,\infty)$ is the linear space of continuous bounded functions on $\mathbb{R}_+$.

The theorem stated below follows from the results of [1–3].

Theorem. *Let all the parameters* a, b^2, n, m, λ, λ_1 *be fixed positive constants, and let the stock reliability condition be satisfied:* $2a/b^2 > 1$. *Then the following assertions hold:*

(I) *The constrained singular nonlocal problem* (3)–(7) *has a unique solution* $\varphi(u)$, *it is a nondecreasing function on* $\mathbb{R}_+$ *and indeed determines the SP in the considered insurance model.*

(II) *As* $u \to +0$, *the behavior of the solution derivatives depends on the relations between the parameters in particular on a sign of the "risk factor"* $i_{\mathrm{r}} = a(m-n) + \lambda_1 n - \lambda m$: (1) *If* $\lambda + \lambda_1 > a$, *then there exists a finite* $\lim_{u\to+0}\varphi'(u) = D_1$; *moreover,* (a) $|\lim_{u\to+0}\varphi''(u)| < \infty$ *if and only if* $\lambda + \lambda_1 > b^2 + 2a$; *more precisely, in this case* $\lim_{u\to+0}\varphi''(u) = D_1D_2 = -D_1\, i_{\mathrm{r}}/[mn(\lambda+\lambda_1-b^2-2a)]$, *so that, if* $D_1 > 0$, *then* $D_2 \le 0$ *for* $i_{\mathrm{r}} \ge 0$ *and* $D_2 > 0$ *for* $i_{\mathrm{r}} < 0$; (b) *if* $\lambda + \lambda_1 \le b^2 + 2a$, *then* $\varphi''(u)$ *is unbounded, but integrable at zero.* (2) *If* $a \ge \lambda + \lambda_1$, *then* $\varphi'(u)$ *is not bounded as* $u \to +0$, *but remains integrable at zero.*

(III) *For large* u, *the solution* $\varphi(u)$ *can be represented as*

$$\varphi(u) = 1 - K\,u^{1-2a/b^2}\,[1+o(1)], \qquad u \to \infty,$$

where $K > 0$ *is a constant (in general the value of* K *cannot be found by local analysis methods).*

(IV) *If* $\lambda + \lambda_1 > b^2 + 2a$ *and* $i_{\mathrm{r}} < 0$, *then* $\varphi'(u)$ *reaches a positive maximum at some point* $u = \widetilde{u} > 0$, *while the solution* $\varphi(u)$ *has an inflection at this point (it is the most risk case).*

The study of this problem demonstrates that investments in risky assets for small and large IS values have opposite effects. For large IS values, the use of risky assets at a constant investment portfolio structure is not favorable from the point of view of survival, while, for small IS values, risky assets are an effective tool for minimizing the overall risk and, hence, for increasing the solvency of the insurer.

References

1. Belkina T.A., Konyukhova N.B., and Kurochkin S.V. Singular boundary value problem for the integrodifferential equation in an insurance model with stochastic premiums: Analysis and numerical solution// Comput. Math. Math. Phys. 2012. V. 52. № 10. P. 1384—1416.
2. Belkina T.A. Risky investment for insurers and sufficiency theorems for the survival probability// Markov Processes Relat. Fields. 2014. V. 20. P. 505–525.
3. Belkina T.A., Konyukhova N.B., and Kurochkin S.V. Dynamical insurance models with investment: Constraint singular problems for integrodifferential equations// Comput. Math. Math. Phys. 2016. V. 56. № 1. P. 47—98.

Risky investments and survival in the dual risk model

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We consider the dual risk model (see, e.g., [1]), where the surplus or equity of a company (in the absence of investments) is of the form

$$R_t = u - ct + \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0. \tag{1}$$

Here R_t is the surplus of a company at time $t \geq 0$; u is the initial surplus, $c > 0$ is the rate of expenses, assumed to be deterministic and fixed; $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of random revenues up to the time t; Z_k $(k = 1, 2, ...)$ are independent identically random variables with a distribution function $F(z)$ $(F(0) = 0$, $\mathbf{E}Z_1 = m < \infty)$ that determine the revenue sizes and are assumed to be independent of $N(t)$.

Let now the whole surplus be continuously invested into risky asset of which price S_t follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t,\ t \geq 0,$$

where μ is the expected return rate, σ is the volatility, B_t is a standard Brownian motion.

Then the resulting surplus process X_t is governed by the equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t + dR_t,\ t \geq 0, \tag{2}$$

with the initial condition $X_0 = u$, where R_t is defined in (1).

Denote $\varphi(u) = P\left(X_t \geq 0,\ \ t \geq 0\right)$ the survival probability (i.e., the probability that bankruptcy will never happen).

The infinitesimal generator $\mathcal{A}$ of the process X_t has the form

$$(\mathcal{A}f)(u) = \frac{1}{2}\sigma^2 u^2 f''(u) + f'(u)[\mu u - c] - \lambda f(u) + \lambda \int_0^\infty f(u+z)\, dF(z),$$

for any function f from a certain subclass of the space $\mathcal{C}^2(\mathbb{R}_+)$ of real-valued, twice continuously differentiable on $(0, \infty)$ functions.

For the case of the exponential revenue sizes, we establish the following statement.

Theorem. *Let* $F(z) = 1 - \exp\left(-z/m\right)$*, all the parameters* μ*,* σ^2*,* m*,* c*,* λ *be fixed positive constants, and let the stock reliability condition be satisfied:* $2\mu/\sigma^2 > 1$*. Then the following assertions hold:*

(I) *the survival probability* $\varphi(u)$ *is the solution to the following singular boundary value problem for the integro-differential equation (IDE) with non-Volterra integral operator:*

$$(\mathcal{A}\varphi)(u) = 0, \quad u > 0, \tag{3}$$

$$\lim_{u \to +0} \varphi(u) = 0, \quad \lim_{u \to \infty} \varphi(u) = 1; \tag{4}$$

(II) *this solution is unique and satisfies the conditions*

$$0 \leq \varphi(u) \leq 1, \quad u \in \mathbb{R}_+,$$

$$0 < \lim_{u \to +0} \varphi'(u) < \infty;$$

(III) *the following asymptotic representations are valid:*

$$\varphi(u) \sim D_1 \left(u + \sum_{k=2}^{\infty} D_k u^k / k \right), \quad u \sim +0,$$

where $D_1 = \varphi'(+0)$, $D_2 = (\mu - \lambda + c/m)/c$,

$$D_3 = \left[D_2(2\mu + \sigma^2 - \lambda + c/m) - \mu/m\right]/(2c),$$

$$D_{k+1} = [D_k(k(k-1)\sigma^2/2 + \mu k - \lambda + c/m)-$$

$$-D_{k-1}((k-2)\sigma^2/(2m) + \mu/m)]/(kc), \quad k = 3, 4, \ldots,$$

and

$$\varphi(u) = 1 - Ku^{1-2\mu/\sigma^2}(1 + o(1)), \quad u \to \infty, \tag{5}$$

where $K > 0$ *is a constant;*

(IV) *as* $u \to +0$*, the behavior of the solution derivatives depends on the relations between the parameters, in particular on a sign of the coefficient* $i_{\mathrm{r}} = (\lambda - \mu)m - c$*: (1) if* $i_r \geq 0$*, then* $\lim_{u\to+0} \varphi''(u) \leq 0$*, moreover, the solution* φ *is concave on* $\mathbb{R}_+$*; (2) if* $i_r < 0$*, then* $\lim_{u\to+0} \varphi''(u) > 0$*, the solution* φ *is convex in a some neighborhood of zero and has an inflexion point.*

For the corresponding results to the classical Cramér-Lundberg risk model, see, e.g., [2]. The asymptotic representation (5) for the survival probability of the process (2) (in the dual risk model) with exponential distribution of the revenue sizes was obtained earlier in [3], where the renewal theory was used to obtain some upper and lower asymptotic bounds for the ruin probability. The regularity of the survival probability was studied in [3] using a method based on integral representations. Note here that the dual model case is rather different from the classical case because the change of two signs to the opposite ones in the equation defining the dynamics of the reserve leads to special technical complications (see [3] in details). We use other approach based on so called sufficiency theorem for the survival probability and the existence theorem for the corresponding singular problems for IDEs (see [4]). This unified approach eliminates need to proof regularity of the survival probability as well as to use its upper and lower bounds. Moreover, the solving of above singular problem for IDE leads to calculation of the survival probability on all non-negative semi-axis. We reduce the problem (3),(4) to a certain initial problem from infinity for some second order ordinary differential equation with respect to the derivative of the survival probability with a normalizing condition. As a result of calculations, we conclude in particular that if the value of safety loading $(\lambda m - c)$ in the model (1) is negative or sufficiently small and the surplus is small too, then the use of the risky investments allows to increase significantly the survival probability.

References

1. Albrecher H., Badescu A., and Landriault D. On the dual risk model with tax payments// Insurance Math. Econom. 2008. V. 42. P. 1086–1094.
2. Belkina T.A., Konyukhova N.B., and Kurochkin S.V. Dynamical insurance models with investment: Constraint singular problems for integro-differential equations// Comput. Math. Math. Phys. 2016. V. 56. № 1. P. 47—98.
3. Kabanov Yu. and Pergamenshchikov S. In the insurance business risky investments are dangerous: the case of negative risk sums// Finance Stochast. (to appear).
4. Belkina T. Risky investment for insurers and sufficiency theorems for the survival probability// Markov Processes Relat. Fields. 2014. V. 20. P. 505–525.

Optimization of marketing strategy of a firm with multiple distribution points of goods

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We present the mathematical model of a firm selling certain product. The feature of this firm is the structure: the firm is divided into several distribution units (for example, department stores) each of which aims to achieve the best sales performance in comparison with other units. Each point has its own marketing budget, approved by the head office, which can not exceed the total marketing budget. The overall aim of the company is «fair» development of all units. Hence, there is the following problem of the budget allocation for all units i in the set A:

$$\begin{cases} pD_i(c) - c_i \to \max\limits_{c_i} \\ \sum_{i\in A} c_i \le C_0 \end{cases}$$

where p is the price of product, $D_i(c)$ and c_i are the demand for product and commercial expenses for unit i accordingly, C_0 is the budget. Thus, there is a kind of competition between units for share of the budget.

The main results of this paper are 1) the proof that there is the unique special solution of described problem and 2) the proof that the problem of «fair» marketing budget allocation is equivalent to the problem of maximizing the total profit:

$$\begin{cases} \sum_{i\in A}(pD_i(c) - c_i) \to \max\limits_{c=(c_i)_{i\in A}} \\ \sum_{i\in A} c_i \le C_0 \end{cases}.$$

References

1. von Heusinger A., Kanzow C. Optimization reformulations of the generalized Nash equilibrium problem using Nikaido–Isoda-type functions // Technical Report, Institute of Mathematics, University of Wurzburg, Wurzburg, 2006.
2. Bass F.M., Krishnamoorthy A., Prasad A., Sethi S.P. Generic and brand advertising strategies in a dynamic duopoly // Marketing Science 24 (4) (2005) 556-568.

Various applications of OR

The two-level model of environmental protection

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The proposed regulating mechanisms use unified and differentiated environmental payments under the presence or absence of quotas and fines control.

Suppose that the regional center may regulate ecological payment rates $p = (p_1, \ldots, p_m)$ (reduction of payment may be a result of offsetting funds or budget exemptions), where p_j is fee for a negative impact on unit volume y_j of j-th pollutant, j=1, …, m. Assume that the volume of the pollutant is proportional to the value of the relevant production factor $y_{ij} = \gamma_{ij} x_i = \sum_{s=1}^{S} \gamma_{ijs} x_{is}$, where $\gamma_{ij} = (\gamma_{ij1}, \ldots, \gamma_{ijs}, \ldots, \gamma_{ijS})$ is the vector of proportional coefficients for j-th pollutant, $\gamma_{is} = (\gamma_{i1s}, \ldots, \gamma_{ijs}, \ldots, \gamma_{ims})$ is the vector of proportional coefficients of all pollutants for i-th enterprise, applying s-th production factor, $x_i = (x_{i1}, \ldots, x_{is}, \ldots, x_{iS})$ is the vector of production factor of i-th enterprise. Let K_i, i=1, …, n, be financial resources of enterprises, $q = (q_1, \ldots, q_S)$ be the vector of prices of production factors (resources). Then the set of control of i-th enterprise is $X_i(p) = \{x_i | Px_i \le K_i,\ x_i \ge 0\}$, i=1, …, n, where

$$P = (q_1 + \sum_{j=1}^{m} p_j \gamma_{ij1},\ \ldots,\ q_s + \sum_{j=1}^{m} p_j \gamma_{ijs},\ \ldots,\ q_S + \sum_{j=1}^{m} p_j \gamma_{ijS}).$$

Output of each enterprise is defined by the vector production function

$f_i(x_i)$, satisfying conditions $f_i(0) = 0$, $\frac{\partial f_i(x_i)}{\partial x_{is}} > 0$, $\xi \frac{\partial^2 f_{ik}(x_i)}{\partial x_i^2} \xi < 0 \ \forall \xi \neq 0$, where $f_{ik}(x_i)$ is k-th component of the vector function $f_i(x_i)$.

If ci is the vector of prices for all products of i-th enterprise, then the problem of maximizing its gross income is

$$G_i(x_i) = A_i f_i(x_i) \to \max_{x_i \in X_i(p)}, \tag{1}$$

Its solution is the optimal strategy of the i-th enterprise $x_i^0(p)$.

Let the center seeks to increase the total gross income of enterprises, i.e. the target function of the center is $F(x_i) = \sum_{i=1}^n \alpha_i G_i(x_i)$, where αi are positive weights, for example, tax payments to the regional budget. It is also assumed that the center interests in a rational use of the region's resources (energy, natural, labor). Then the problem of the center is

$$F(x^0(p)) = \sum_{i=1}^n \alpha_i G_i(x_i^0(p)) \to \max_{p | \sum_{i=1}^n x_i^0(p_i) \leq X}, \tag{2}$$

where X is the limit of resources amount. The solution of problem (2) gives the optimal strategy of the center p^0.

Let's consider the problem of centralized scheme control

$$F(x) = \sum_{i=1}^n \alpha_i G_i(x_i) \to \max_{x | \sum_{i=1}^n x_i \leq X}, \tag{3}$$

its solution is vector $x_i^* = (x_{i1}^*, \ \dots, \ x_{is}^*, \ \dots, \ x_{iS}^*)$.

We introduce the Lagrange function for problem (3) $L(x, \mu) = = \sum_{i=1}^n \alpha_i G_i(x_i) + \mu(X - \sum_{i=1}^n x_i)$, where $\mu = (\mu_1, \ \dots, \ \mu_S)$ is the vector Lagrange multiplier, and consider for i-th element of lower-level the system of linear equations with unknown k_i, $p_i = (p_{i1}, \ \dots, \ p_{im})$:

$$k_i \mu_s = q_s + \sum_{j=1}^m p_{ij} \gamma_{ijs}, \ s = 1, \ \dots, \ S, \quad K_i = k_i \mu x_i^*. \tag{4}$$

Denote p_{0i} environmental payments vector for i-th enterprise, defined by legislation.

Theorem 1. Let functions $G_i(x_i)$, *i=1*, $\dots$, n, be continuous, strictly concave with respect to all their variables, and have continuous positive derivatives with respect to x_{is}, the system of linear equations (4) has positive solution such that $p_i \leq p_{0i}$, $i = 1, \ \dots, \ n$. Then by choosing differentiated environmental payments pi for lower-level elements in problem (2) the center provides the global maximum of its criterion, i.e. achieves perfect reconciliation of interests.

Assume that the center has the ability to assign only unified environmental p and additionally permissible levels of pollution (quotas) and fines for exceeding these quotas. The amount of fines z_{ij} per unit for the excess of j-th type of pollution and quotas $\beta_i = (\beta_{i1}, \ldots, \beta_{ij}, \ldots, \beta_{im})$ determined by the center for each enterprise satisfy conditions $z_{ij} \geq 0$, $\beta_i \geq 0, i=1, \ldots, n, \sum_{i=1}^{n} \beta_{ij} = \mathrm{B}_j$, where B_j – is fixed value, means the maximum permissible level of pollution by j-th indicator for the whole region. Denote $z_i = (z_{i1}, \ldots, z_{im}), z = (z_1, \ldots, z_n), \beta = (\beta_1, \ldots, \beta_i, \ldots, \beta_n)$. The target function of the center is $F(x) = \sum_{i=1}^{n} \alpha_i G_i(x_i)$.

As a fine function we take the total excess on all types of pollution. Then the problem of i-th enterprise is

$$A_i f_i(x_i) \to \max_{x_i \in X_i'(p, z_i, \beta_i)}, \tag{5}$$

$$X_i'(p, z_i, \beta_i) = \{x_i | P x_i + \sum_{j=1}^{m} z_{ij} \max(0, \gamma_{ij} x_i - \beta_{ij}) \leq K_i, x_i \geq 0\}.$$

We introduce the vector of the maximum permissible levels exceeds $w_i = (w_{i1}, \ldots, w_{im})$. Then problem (5) takes form

$$G_i(x_i) = A_i f_i(x_i) \to \max_{(x_i, w_i) \in X_i(p, z_i, \beta_i)}, \tag{6}$$

$$X_i(p, z_i, \beta_i) = \{(x_i, w_i) \geq 0 | \gamma_{ij} x_i - \beta_{ij} \leq w_{ij},$$

$P x_i + \sum_{j=1}^{m} z_{ij} w_{ij} \leq K_i,\ j = 1, ..., m\}$. Let $x_i^0(p, z_i, \beta_i)$ be the solution of problem (6). The problem of the center optimal control is

$$\sum_{i=1}^{n} \alpha_i G_i(x_i^0(p, z_i, \beta_i)) \to \max_{(p, z, \beta) \in Q}, \tag{7}$$

$Q = \{(p, z, \beta) \geq 0 | \sum_{i=1}^{n} \beta_{ij} = \mathrm{B}_j, j = 1, ..., m, \sum_{i=1}^{n} x_i^0(p, z_i, \beta_i) \leq X\}$. Denote the center optimal control (p^0, z^0, β^0).

We introduce the Lagrange function for problem (7) $\tilde{L}_i(x_i, w_i, \lambda_{i1}, \lambda_{i2}) = G_i(x_i, w_i, p, z_i, \beta_i) + \lambda_{i1}(K_i - P x_i - \sum_{j=1}^{m} z_{ij} w_{ij}) + \sum_{j=1}^{m} \lambda_{ij2}(w_{ij} + \beta_{ij} - \gamma_{ij} x_i)$, where $\lambda_{i1} \geq 0$, $\lambda_{i2} \geq 0$ are Lagrange multipliers, λ_{i2} is m-dimensional vector.

The problem of centralized control has the form

$$\sum_{i=1}^{n} \alpha_i G_i(x_i)) \to \max_{x \in Q_1}, \tag{8}$$

$$Q_1 = \{x| \sum_{i=1}^{n} \gamma_{ij} x_i \leq \mathrm{B}_j, j = 1, \ ..., \ m, \ \sum_{i=1}^{n} x_i \leq X\}.$$

Denote the solution of problem (8) by $x_i^* = (x_{i1}^*, \ \dots, \ x_{is}^*, \ \dots, \ x_{iS}^*)$. Consider the system of equations:

$$\begin{array}{l} \lambda_{i1} P x_i^* + \sum_{j=1}^{m} \lambda_{ij2}(\beta_{ij} - \gamma_{ij} x_i^*) = \lambda_{i1} K_i, \quad \lambda_{i1} P_s = \mu_{1s}/\alpha_i, \\ \sum_{j=1}^{m} \lambda_{ij2} \gamma_{ijs} = (\sum_{j=1}^{m} \mu_{2j} \sum_{i=1}^{n} \gamma_{ijs})/\alpha_i = 0, \\ \quad i = 1, \ ..., \ n, \ s = 1, \ ..., \ S. \end{array} \tag{9}$$

Denote the fixed vector of utmost environmental payments, defined by the legislation, by p_0.

Theorem 2. Let functions $G_i(x_i)$, i=*1*, ..., n, be continuous, strictly concave with respect to all their variables, and have continuous positive derivatives with respect to *xis*, the system of linear equations (9) has positive solution λ_1, λ_{i2}, p, β such that $p \leq p_0$. Then by choosing unified environmental payments p, quotas β and fines z and for lower-level elements in problem (9) the center provides the global maximum of its criterion, i.e. achieves perfect reconciliation of interests.

Dynamic model of collective decision making

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In this work the already constructed in [2] model is generalized on continuous time case and applied to some elementary examples. In [1],[2] the author speaks about model, describing process of collective decision making. He fixed one state and crowd of people; everybody from this crowd can go to this state or can remain. This state we will call main state. Everybody has his own opinion about switch to this state generated before communication with other people. This opinion will estimated by $\alpha-$probability of preparedness to go to the main state. After communication α will change. This new probability we will call p. Also everybody has his own characteristics describing his individual features. It means that member numbered i is described by:

- μ_i-probability of independent decision making;
- $\lambda_{ij}-$probability of following member numbered j in decision making;

- α_i;
- p_i.

Using this parameters author created the main system of equations:

$$p_i = \mu_i \alpha_i + (1 - \mu_i) \sum_{j=1}^{N} \lambda_{ij} p_j, \quad i = 1, .., N. \tag{1}$$

Parameters λ_{ij} are bounded by:

$$\sum_{j=1}^{N} \lambda_{ij} = 1, \quad \lambda_{ii} = 0, \quad i = 1, ..N. \tag{2}$$

Here p_i are variables. System (1) is generalized on continuous time case:

$$\frac{dp_i(t)}{dt} = \mu_i(\alpha_i - p_i(t)) + (1 - \mu_i) \sum_{j=1}^{N} \lambda_{ij} (p_j(t) - p_i(t)), \quad i = 1, .., N. \tag{3}$$

In system (3) α_i are initial conditions and $p_i(t)$ are changing during the time. So we get Cauchy problem. This Cauchy problem has a unique correct ($p_i(t) \in [0, 1]$) solution. If $\mu_i > 0 \ \forall i$, time-independent solutions of this system (3) are asymptotically stable. (3) is applied in quite ordinary case: when crowd of people can be separated on three groups. First has negative opinion about main problem, second has positive opinion about main problem. The rest of people are not sure. As a result I got solutions quite good coordinated with reality.

References

1. Krasnoshekov P.S., Petrov A.A. Principles of the construction of models // M.: Fasis, 2000.
2. Krasnoshekov P.S., The simplest mathematical model behavior. The psychology of conformism // Mathematical modeling. 1998. 10(7). P. 76–92.

Socialization as an effective mechanism of strategy alteration from individual to cooperative: some psychophysiological aspects*

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The classical economic theory suggests that economic agents are rational, i.e. they make decisions according the maximization of their own profit. The experimental economics allows checking the validity of this statement in laboratory conditions. People evade rational strategies in some situations and choose the ones that lead to less profit at this particular moment, but have the perspective benefit for the society in general. These strategies we call cooperative, prosocial and leading to equality. Since the choice of cooperative strategies contradicts the rational choice theory the question arises: what motivates some people still follow the cooperative or prosocial strategies? An important question that still remains: how can we accomplish the strategy alteration from individual and rational to cooperative and prosocial?

It is known from the social psychology that not all decisions are made according to the expected future reward. There is some moral satisfaction from the fact that the trust is established in group and all participants receive the same payoff. Thus, we assume that the utility function depends on the social component that covers the dissatisfaction of receiving fewer benefits.

The MIPT Experimental Economics Laboratory and Skoltech are used to carry out all experiments. The treatments comprise knowledge from experimental economics and social psychology [1]. Each experiment consists of a different set of 12 students, pre-selected before the experiment to be unfamiliar with one another. In the laboratory, we studied the nature of such social qualities of a person as cooperativness, fairness, trust, gratitude in the groups socialized differently. We used the following 2x2 games: Prisoners' Dilemma, Ultimatum Game, and Trust Game. The research goal is to find and study the mechanism that effectively alter the participants strategies from individual to cooperative without using social or material punishments. In course of our studying we discovered such a mechanism - a group socialization.

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Each experiment is divided into 3 consecutive phases: anonymous game phase in group of 12, socialization phase, and socialized game phase in group of 6 or 4. We use different variants of socialization. However we always include the introduction step and division participants on two or three equal groups and some teamwork in newly formed groups. The game phases consist of a number of periods in a randomly formed pairs. On the first phase pairs form from the total sample of participants, on the third phase pairs form within newly formed groups.

To study changes in people's attitudes after the socialization we use an interdisciplinary approach combining methods from experimental economics, social psychology and psychophysiology. During the laboratory experiment we measure the stabilograms [2] and RR-intervals of all participants. These data are compared with each other, with the behavioral characteristics and data from psychological tests [3-4].

Results.

1. Socialization promotes alteration of the participants' strategies from individual to cooperative.

2. The effect of socialization is different between sexes.

The initial (before socialization) level of cooperation among females is equal or higher than among males (on average $\delta = .02$, $N_m = 202, N_f = 122$, wilcoxon-test, p-value = .05). Whereas after the socialization the percentage of cooperation among males is higher than among females (on average $\delta = .15$, $N_m = 202, N_f = 122$, wilcoxon-test, p-value = .001).

3. The psychological type effects the change of social indicators after socialization. We find psychological types with the highest percentage of the transition from individual strategies before to cooperative strategies after socialization.

4. The relationship between energy and entropy of the participants during an economic experiment, stress levels (an indicator, which can be derived from measurements of RR-intervals) and psychological personality type is established.

References

1. Berkman E.T., Lukinova E., Menshikov I., Myagkov M. Sociality as a Natural Mechanism of Public Goods Provision. PLoS ONE, 10(3), 2015, e0119685.
2. Menshikov I.S. Laboratory analuses of the context influences on the decision making. // Proceedings of MIPT-2014.-6(4), pp 67-77.

3. Menshikova O.R., Menshikov I.S., Sedush A.O. Laboratory studies of the differences in the behavior of men and women before and after socialization. Oxford Journal of Scientific Research, 2015, No.1. (9) (January-June). Volume III. "Oxford University Press 2015, pp. 339-346.
4. Menshikova O.R., Menshikov I.S., Sedush A.O. Influence of three types of socialization on the behavior of men and women in social and economic experiments. Proceedings of MIPT, 2015, pp 56-65.

The asymptotic solution of a singularly perturbed initial boundary value problem

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This work involves the problems of solving the tasks arising in the study and description of the processes occurring in the laser targets [1]. An understanding of such processes makes possible the implementation and control of technological procedures of thermo-nuclear synthesis from the creation of laser target to their delivery to the place of ignition and management the launch of a thermonuclear reaction. The following is a mathematical model of the single-layer shells filling with gas, which is reduced to linear singularly perturbed initial-boundary value problem of parabolic type [2]. Processes such as cooling of the target and the problem of degradation of the fuel layer by heating the target in the reactor chamber by electromagnetic radiation [3] are reduced to a similar class of problems.

Below we going to state the initial boundary value problem for the function $u(x,t),\ x \in [0,1],\ t > 0$, that satisfies the parabolic equation

$$\varepsilon \frac{\partial u}{\partial t} = \frac{1}{(1-\delta x)^2} \frac{\partial}{\partial x} (1-\delta x)^2 \frac{\partial u}{\partial x}. \tag{1}$$

When formulating the problem the boundary conditions are one of the most important factor. Let us when $x = 1$

$$u(1,t) = \mu(t), u(1,0) = \mu(0) = b, \tag{2}$$

b - determined value and μ - unknown function, which satisfies the following ordinary differential equation

$$\frac{d\mu}{dt} = -\alpha \frac{\partial u(x,t)}{\partial x} \left|_{x=1}\right. ,\ \alpha > 0. \tag{3}$$

When $x = 0$ let

$$u(0,t) = \gamma \cdot \mu(t) + f(t),\ \gamma = const \geq 0 \tag{4}$$

here $f(t)$ - determined time function. Initial conditions are determined by next statement

$$u(x,0) = U(x), \tag{5}$$

compatibility conditions are:

$$U(1) = b\ , U(0) = \gamma b + f(0). \tag{6}$$

Using the work [4] research methods we are getting the above stated problem decision in the following theorem form.

Theorem. *The initial boundary value problem (1)-(6) can be solved and the solution can be described as follows*

$$u(x,t) = \left(\gamma + \frac{\kappa x}{1-\delta x}\right)\mu(t) + \frac{(1-x)}{(1-\delta x)}f(t) +$$

$$+\frac{1}{(1-\delta x)}\left[v_s\left(x, \frac{t}{\varepsilon}\right) + \varepsilon w\left(x, \frac{t}{\varepsilon}\right)\right],$$

where

$$v_s\left(x, \frac{t}{\varepsilon}\right) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\pi^2 n^2 t}{\varepsilon}\right)\sin(\pi n x),$$

and

$$\mu(t) = \mu_0(t) + \varepsilon M\left(\frac{t}{\varepsilon}, \varepsilon\right) = e^{-\beta_1 t}\left(b + \beta\int_0^t e^{\beta_1 s} f(s)ds\right) b + \varepsilon M\left(\frac{t}{\varepsilon}, \varepsilon\right).$$

Note that functions M, w are uniformly bounded and the initial conditions discrepancy is compensated with function $v_s\left(x, \frac{t}{\varepsilon}\right)$ and is quickly decreased to zero while t increasing.

In conclusion, we should note one significant fact that values of parameter γ define the process a) $\gamma = 1$ filling of target with gas; b) $\gamma = 0$ the process of cooling the gas inside the target.

References

1. Aleksandrova I.V., Belolipetskii A.A., Koresheva E.R. Current inertial thermonuclear synthesis program and state of cryogenic fuel targets problem // The journal "Bulletin of the Russian Academy of Natural Sciences 2007. № 2. P. 15–20.
2. Aleksandrova I.V., Belolipetskii A.A. Mathematical models for filling polymer shells with a real gas fuel. // Laser and Particle Beams, 1999. Vol. 17, № 4. P. 701–712.
3. Belolipetskii A.A., Malinina E.A., Semenov K.O. Mathematical model of fuel layer degradation when the laser target is heated by thermal radiation in the reactor working chamber // Computational Mathematics and Modeling,2010. Vol. 21, № 1. P. 1–17.
4. Belolipetskii A.A., Ter-Krikorov A.M. The solution of a singularly perturbed initial-boundary value problems for linear parabolic equations //Works of MIPT, 2011. Vol. 3, № 1. P. 14–17.

Game-theoretic models

Search numbers on graphs of block structure

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We consider a problem of discrete graph searching. Invisible fugitive, whose movements are unpredictable, moves on graph. There is a set of searchers, whose goal is to find the fugitive. The conditions of capture fugitive depend on type of search. In each case finding the minimum k, such that k searchers can capture any fugitive in graph G, is the goal. This minimum k is called the *search number* of graph G.

This problem can be formulated differently. The edge is *clear* if it is guaranteed no fugitive on this edge, else the edge is *contaminated*. Initially all graph's edges are contaminated and the searchers' goal is to clear all graph's edges. There are three possible *search steps*: to place a searcher on a node, to remove a searcher from a node and to move a searcher along an edge. A sequence of search steps that results in all edges being clear, is a *search strategy*. A strategy is *monotone* if no recontamination ever occurs. If the set of clear edges always induces a connected subgraph, a search strategy is *connected*. Connected search simulates a situation, when searchers want to have a safe transmission channel.

Three types of search are considered: *edge search*, *mixed search* and *connected mixed search*. Their search numbers are denoted by *s(G), mixs(G), cmixs(G)*. The first formulation of edge searching problem was given by N.N. Petrov in [1] and T. Parsons in [2]. A searcher must traverse the edge from one end–point to the other to clear the edge. The conditions of clearing in mixed search consist of the condition in edge search and a new opportunity, which is to place searchers on the both edge's end–points. A clear edge $e = (u_1, u_2)$ is preserved from recontamination if one of these statements is true for $u_i,\ i = 1, 2$: either searcher remains in u_i, or all other edges incident to u_i are clear. In other words, a clear edge e is recontaminated if there exists a path between e and contaminated edge with no searcher on any node of the path.

Graph searching problems are attractive for their correspondence with classical *width–parametres*, serving as a model for important applied problems, which were described in [3], [4]. Connection between graph searching and *pebbling* was found in [5]. The relationships between search numbers was showed in [6]. Let's mention several of them:

- $mixs(G) - 1 \leqslant pw(G) \leqslant mixs(G)$
- $mixs(G) \leqslant s(G) \leqslant mixs(G) + 1$
- $mixs(G) \leqslant cmixs(G)$

It is known, that it is enough to consider monotone strategies if the goal is to find *s(G), mixs(G)*. In most cases, the class of graphs that can be cleared by the edge search strategy using at most k searchers is minor closed. This fact is true for the mixed search, too. In case of the connected mixed search, there is a counterexample given in [7] and it is proven that the class of graphs that can be cleared by the connected strategy using at most k searchers is not minor closed.

We introduce a special class of graphs to research connection between the search numbers $s(G)$, $mixs(G)$, $cmixs(G)$. We propose definition of a *block* $m \times n$. It is a graph that can be imagined such as a grid $m \times n$, where m is the number of rows and n is the number of columns. A block have a *boundary*, which is the subgraph induced with the set of all vertices of degree less than 4. The boundary is divided into four parts (left, right, top and bottom) intuitively. For any block G (size $m \times n$) we show that $s(G) = mixs(G) + 1 = cmixs(G) + 1 = \min\{m, n\} + 1$. Then we introduce an operation with two blocks B_1 and B_2 and call it by *gluing*. This operation means that all vertices of one boundary's part of a block B_1 are merged with vertices of one boundary's part of a block B_2. Gluing of B_1 and B_2 is denoted by $B_1 \sqcup B_2$. Also we can define a boundary of $B_1 \sqcup B_2$ such as a subgraph, which contains boundaries of B_1 and B_2 except merged vertices whose degree was 3 in $B_1,\ B_2$. The gluing is intuitively generalized for any amount of blocks. Resulting graphs are called *graphs of block structure*. Now we can introduce a new *block search* on graphs of block structure. For block search only strategies, which have a following property, are considered: on every steps there exists no more than one block, that have both clear and contaminated edges except boundary's edges. The conditions of clearing are equal to mixed search. A *block search number* is denoted by $bs(G)$.

We research gluing of two blocks B_1 $(m_1 \times n_1)$ and B_2 $(m_2 \times n_2)$. The first class of resulting graphs contains all graphs $B_1 \sqcup B_2$, when the pair of vertices of degree two is merged. Let $m_2 \geqslant m_1$ and all vertices of right boundary of B_1 are merged with vertices of left boundary of B_2. For all graphs in this class it is showed that $bs(G) = mixs(G) = cmixs(G) = \min\{\max\{m_1, n_2\}, m_2, n_1 + n_2 - 1\}$. The second class contains the other graphs $B_1 \sqcup B_2$. Without loss of generality, we assume that all vertices of bottom boundary of B_2 are merged with vertices of top boundary of B_1. Let $n_2 < n_1$, then $n_1 = k - 1 + n_2 + p - 1$, where $p - 1$, $k - 1$ are amount of top boundary's vertices of B_1, which are situated left and right of merged vertices, and $p \geqslant k$. In this case we show that $bs(G) = mixs(G) = cmixs(G) = \min\{\max\{2m_1, n_2\}, \max\{m_1, n_2 + k - 1\}, n_1, m_1 + m_2 - 1, m_1 + n_2\}$.

Further we consider operation deletion of the internal edges and vertices of block B_1 $(m_1 \times n_1)$ from block B_2 $(m_2 \times n_2)$, where $m_2 > m_1$, $n_2 > n_1$. For all resulting graph it is showed that $mixs(G) = cmixs(G) = \min\{M + m, m_2, n_2\}$, where m is the minimum number of vertices in a row (left and right parts) or in a column (bottom and top parts) from boundary of deleted block B_1 to boundary of block B_2, M is the maximum number of such vertices.

References

1. Petrov N.N. A problem of pursuit in the absence of information on the pursued // Differential Equations. 1982. V. 18, № 8, P. 1345–1352.
2. Parsons T.D. Pursuit–evasion in a graph // Theory and applications of graphs. Berlin: Springer. 1978. P. 426–441.
3. Abramovskaya T.V., Petrov N.N. The theory of guaranteed search on graphs // Differential Equations and Control Processes. 2012. № 2 P. 9–65.
4. Fomin F.V., Thilikos D.M. An annotated bibliography on guaranteed graph searching // Theoretical Computer Science. 2008. V. 399, № 3. P. 236–245.
5. Kirousis L.M., Papadimitriou C.H. Searching and pebbling // Theoretical Computer Science. 1986. V. 47, № 1. P. 205–218.

6. Boting Yang Strong-mixed Searching and Pathwidth // Journal of Combinatorial Optimization. 2007. V. 13, № 1, P. 47–59.
7. Barriere L., Fraigniaud P., Santoro N., Thilikos D.M. Connected and Internal Graph Searching // In 29th Workshop on Graph Theoretic Concepts (WG). Springer-Verlag. 2003. P. 34–45.

Generalization of binomial coefficients to numbers on the nodes of graphs*

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The topic of this work does not relate directly to game theory, but the interest for this study is strongly influenced by our study of Shapley-type solution concepts for cooperative games with limited cooperation introduced by means of communication graphs. If there are no restrictions on cooperation, the classical Shapley value assigns to each player as a payoff the average of the players' marginal contributions with respect to all possible orderings of the players. However, in case of limited cooperation represented by a graph not all orderings of the players are feasible, but only those that are consistent with the graph. When the graph is a line-graph, the numbers of feasible orderings starting from each of its nodes are given by the binomial coefficients.

The triangular array of binomial coefficients, or Pascal's triangle, is formed by starting with an apex of 1. Every row of Pascal's triangle can be seen as a line-graph, to each node of which the corresponding binomial coefficient is assigned. We show that the binomial coefficient of a node is equal to the number of ways the line-graph can be constructed when starting with this node and adding subsequently neighboring nodes one by one. Using this interpretation we generalize the sequences of binomial coefficients on each row of Pascal's triangle to so-called Pascal graph numbers assigned to the nodes of an arbitrary (connected) graph. We show that on the class of connected cycle-free graphs the Pascal graph numbers have properties that are very similar to the properties of

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binomial coefficients. We also show that for a given connected cycle-free graph the Pascal graph numbers, when normalized to sum up to one, are equal to the steady state probabilities of some Markov process on the nodes. Properties of the Pascal graph numbers for arbitrary connected graphs are also discussed. Because the Pascal graph number of a node in a connected graph is defined as the number of ways the graph can be constructed by a sequence of increasing connected subgraphs starting from this node, the Pascal graph numbers can be seen as a measure of centrality in the graph.

Controlled dynamics in multicriteria optimization*

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A mathematical model of terminal control with two basic components: a controlled dynamics and a boundary value problem in the form of multicriteria equilibrium model, is considered. The boundary value problem describes a controlled object situated in a equilibrium state. Under the influence of external disturbances the object loses its state of stability and must be returned to equilibrium. The saddle point approach was used to do this, and the extraproximal method was applied to find a solution. The convergence of the method to solution was proved.

Boundary value problem. A group of m participating countries creates a community for the realization of some economic project. It is assumed that by the time of the community creation, the member countries have already identified their interests and objectives in the project, set types and amount of resources required to participate in integration. Interests of each of the participants are described by cost objective functions $f_i(x_1)$, $i = \overline{1,m}$, which are defined on a common set of resources $X_1 \subseteq \mathrm{R}^n$. Each of participants wants to minimize the cost of its contribution to the overall project. In the first approximation, this situation can be described as a simple multicriteria optimization problem:

$$f(x_1^*) \in \mathrm{ParetoMin}\{f(x_1) \mid x_1 \in X_1\}, \tag{1}$$

where $f(x_1) = (f_1(x_1), f_2(x_1), ..., f_m(x_1))$ is a vector criterion; convex scalar function $f_i(x_1)$ is value of resources that must be entered in the

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community by i-th participant to implement the project. The problem (1) generates a set of solutions in the form of vast variety of Pareto-optimal points.

Along with the individual interests of participants there exist also group interests, for example, the cost of the whole project. For different Pareto-optimal estimates this cost is different. It is natural to choose the project with a minimum value. Thus, it is necessary to formulate a mathematical model that takes into account both the individual interests of each participant and group (collective) interests of the community. As a result, the following two-person game with Nash equilibrium was proposed [1]:

$$\langle \lambda^*, f(x_1^*)\rangle \in \mathrm{Min}\{\langle \lambda^*, f(x_1)\rangle \mid x_1 \in X_1\}, \tag{2}$$

$$\langle \lambda - \lambda^*, f(x_1^*) - \lambda^*\rangle \leq 0, \quad \lambda \geq 0. \tag{3}$$

Formulation of terminal control problem. We add a controlled dynamics to the problem (2),(3) and formulate the following common dynamic model with multicriteria optimization boundary value problem:

$$\frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \;\; t_0 \leq t \leq t_1, \; x(t_0) = x_0, \tag{4}$$

$$x(t_1) = x_1^* \in X_1 \subseteq \mathrm{R}^n, \; u(\cdot) \in \mathrm{U}, \tag{5}$$

$$\mathrm{U} = \{u(\cdot) \in \mathrm{L}_2^r[t_0, t_1] \mid \|u(\cdot)\|^2_{\mathrm{L}_2^r} \leq \mathrm{C}\}, \tag{6}$$

where x_1^* is x_1-component of solution for multicriteria equilibrium problem (2),(3). Here $D(t), B(t)$ are continuous matrices, x_0 is initial condition, $x(t) \in \mathrm{AC}_2^n[t_0, t_1]$ (linear variety of absolutely continuous functions). The dynamic model (2)-(6) describes the transition of controlled object from the initial state x_0 to a terminal state $x(t_1) = x_1^*$, which is given implicitly as the solution of (2),(3). We look for a control $u^*(t) \in \mathrm{U}$ such that the trajectory $x^*(t)$ has got by its right end to the appropriate component $x^*(t_1)$ of boundary value problem's solution.

Saddle point approach to the problem. We associate the problem (2)-(6) with the saddle-point-type function, which will play a role similar to the Lagrange function in convex programming:

$$\mathcal{L}(\lambda, \psi(t); x_1, x(t), u(t)) =$$

$$= \langle \lambda, f(x_1) - \frac{1}{2}\lambda\rangle + \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt, \tag{7}$$

defined for all $(\lambda, \psi(\cdot)) \in \mathrm{R}^m_+ \times \Psi^n_2[t_0, t_1]$, $(x_1, x(t), u(t)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$. In the case of regular constraints, the function (7) always has a saddle point $(\lambda^*_1, \psi^*(\cdot); x^*_1, x^*(\cdot), u^*(\cdot))$, which is the solution of the problem. Therefore, the problem (2)-(6) is reduced to finding the saddle points of (7).

Method to solve the problem. The dual extraproximal method that guarantees the convergence to the solution of saddle point problem (2)-(6), has been applied [1]:

$$\bar{\lambda}^k = \operatorname{argmin}\left\{\frac{1}{2}|\lambda - \lambda^k|^2 - \alpha\langle\lambda, f(x^k_1) - \frac{1}{2}\lambda\rangle \mid \lambda \geq 0\right\}, \tag{8}$$

$$\bar{\psi}^k(t) = \psi^k(t) + \alpha\left(D(t)x^k(t) + B(t)u^k(t) - \frac{d}{dt}x^k(t)\right), \tag{9}$$

$$(x^{k+1}_1, x^{k+1}(\cdot), u^{k+1}(\cdot)) = \operatorname{argmin}\left\{\frac{1}{2}|x_1 - x^k_1|^2 + \right.$$

$$+\alpha\langle\bar{\lambda}^k, f(x_1) - \frac{1}{2}\bar{\lambda}^k\rangle + \frac{1}{2}\|x(t) - x^k(t)\|^2 + \frac{1}{2}\|u(t) - u^k(t)\|^2 +$$

$$\left. + \alpha\int_{t_0}^{t_1}\langle\bar{\psi}^k(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt\right\}, \tag{10}$$

$$\lambda^{k+1} = \operatorname{argmin}\left\{\frac{1}{2}|\lambda - \lambda^k|^2 - \alpha\langle\lambda, f(x^{k+1}_1) - \frac{1}{2}\lambda\rangle \mid \lambda \geq 0\right\}, \tag{11}$$

$$\psi^{k+1}(t) = \psi^k(t) + \alpha\left(D(t)x^{k+1}(t) + B(t)u^{k+1}(t) - \frac{d}{dt}x^{k+1}(t)\right), \ \alpha > 0, \tag{12}$$

where a minimum in (13) is computed in all $(x_1, x(\cdot), u(\cdot)) \in X_1 \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$. A similar approach was considered in [2].

Theorem (on convergence of the method). *If the solution of equilibrium problem* (2)-(6) *exists, functions* $f_i(x_1)$, $i = \overline{1, m}$, *are convex and subject to Lipschitz condition with constant* L, *then the sequence generated by the dual extraproximal method* (8)-(12) *with the parameter* α, *satisfying the condition* $0 < \alpha < \alpha_0$, *where* α_0 *is a defined constant, contains a subsequence that converges to one of the solutions* $(\lambda^*, \psi^*(\cdot); x^*_1, x^*(\cdot), u^*(\cdot))$ *of the problem. In this case, the convergence in controls is weak, the convergences in phase and conjugate trajectories (as well as in terminal variables) are strong.*

References

1. Antipin A.S., Khoroshilova E.V. Multicriteria boundary value problem in dynamics // Trudy Instituta matematiki i mekhaniki UrO RAN. Yekaterinburg, 2015. V. 21, № 3. P. 20–29 (in Russian)
2. Khoroshilova E.V. Extragradient-type method for optimal control problem with linear constraints and convex objective function // Optimization Letters. Springer Verlag, 2013. V. 7, № 6. P. 1193–1214.

On a construction generating potential games*

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Strategic games are considered where each player's total utility is the sum of local utilities obtained from the use of certain "facilities." All players using a facility obtain the same utility therefrom, which may depend on the identities of users and on their behavior. If a regularity condition is satisfied by every facility, then the game admits an exact potential [1]; both congestion games [2] and games with structured utilities [3], as well as games of social interactions considered in [4], are included in the class and satisfy that condition. Under additional assumptions the potential attains its maximum, which is a Nash equilibrium of the game.

A *strategic game* Γ is defined by a finite set N of players, and, for each $i \in N$, a set X_i of *strategies* and a real-valued *utility function* u_i on the set $X_N := \prod_{i\in N} X_i$ of *strategy profiles.* We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ and $X_I := \prod_{i\in I} X_i$ for each $I \in \mathcal{N}$.

A function $P : X_N \to \mathbb{R}$ is an *exact potential* of Γ if

$$u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N)$$

whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$. If $x_N^0 \in X_N$ maximizes P over X_N, then, obviously, x_N^0 is a Nash equilibrium.

A *game with additive common local utilities* (an *ACLU game*) may have an arbitrary finite set N of players and arbitrary sets of *strategies* X_i

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$(i \in N)$, whereas the utilities are defined by the following construction. First of all, there is a set A of *facilities*; we denote $\mathcal{B}$ the set of all (nonempty) finite subsets of A. For each $i \in N$, there is a mapping $\mathrm{B}_i \colon X_i \to \mathcal{B}$ describing what combination of facilities player i uses when choosing x_i. Every strategy profile x_N determines *local utilities* at all facilities $\alpha \in \mathrm{A}$; each player's *total utility* is the sum of local utilities over chosen facilities. The exact definitions need plenty of notations.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^- := \{i \in N \mid \forall x_i \in X_i\,[\alpha \in \mathrm{B}_i(x_i)]\}$ and $I_\alpha^+ := \{i \in N \mid \exists x_i \in X_i\,[\alpha \in \mathrm{B}_i(x_i)]\}$; without restricting generality, we may assume $I_\alpha^+ \neq \emptyset$. For each $i \in I_\alpha^+$, we denote $X_i^\alpha := \{x_i \in X_i \mid \alpha \in \mathrm{B}_i(x_i)\}$. Then we set $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I\rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I \in X_I^\alpha\}$. The local utility function at $\alpha \in \mathrm{A}$ is $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$. For every $\alpha \in \mathrm{A}$ and $x_N \in X_N$, we denote $I(\alpha, x_N) := \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\} \in \mathcal{I}_\alpha$. The total utility function of each player i is

$$u_i(x_N) := \sum_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha, x_N)}).$$

We call a facility $\alpha \in \mathrm{A}$ *regular* if there is a real-valued function $\psi_\alpha(\cdot)$ defined for integer m between $\max\{1, \#I_\alpha^-\}$ and $\#I_\alpha^+ - 1$ such that $\varphi_\alpha(I, x_I) = \psi_\alpha(\#I)$ whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I \in X_I^\alpha$.

In other words: whenever a regular facility α is not used by all potential users, neither the list of users, nor their strategies matter, only the number of users.

We call an ACLU game *regular* if so is every facility. Both congestion games and games with structured utilities are regular ACLU games.

Theorem 1. Every regular ACLU game admits an exact potential.

Let a finite set N of players be fixed. An *autonomous facility* α is defined by two subsets $I_\alpha^- \subseteq I_\alpha^+ \in \mathcal{N}$ [I_α^- may be empty], a set X_i^α of relevant strategies for each $i \in I_\alpha^+$, and a local utility function $\varphi_\alpha \colon \Xi^\alpha \to \mathbb{R}$, where $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi^\alpha := \{\langle I, x_I^\alpha\rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I^\alpha \in X_I^\alpha\}$, exactly as above. We call an autonomous facility α *regular* if it satisfies the same condition.

Let α be an autonomous facility, and let Γ be an ACLU game with the same set N, a finite set A such that $\alpha \notin \mathrm{A}$, and $X_i \cap X_i^\alpha = \emptyset$ for each $i \in N$. An *extension of* Γ *with* α is a strategic game Γ^* satisfying these conditions: $N^* = N$; $\mathrm{A}^* = \mathrm{A} \cup \{\alpha\}$; for each $i \in N$, $X_i^* = X_i \cup X_i^\alpha$ if $i \in I_\alpha^+$ and $X_i^* := X_i$ otherwise, $\mathrm{B}_i^*(x_i) = \mathrm{B}_i(x_i)$ for each $x_i \in X_i$, and, for each $x_i^\alpha \in X_i^\alpha$, there is $\sigma_i(x_i^\alpha) \in X_i$ such that $\mathrm{B}_i^*(x_i^\alpha) = \{\alpha\} \cup \mathrm{B}_i(\sigma_i(x_i^\alpha))$;

whenever $I \in \mathcal{I}_\alpha$ and $x_I^\alpha \in X_I^\alpha$, there holds $\varphi_\alpha^*(I, x_I^\alpha) = \varphi_\alpha(I, x_I^\alpha)$; whenever $\beta \in \mathrm{A}$, $I \in \mathcal{I}_\beta$, $x_I \in X_I^{*\beta}$, and $J := \{i \in I \mid x_i \in X_i^\alpha\}$, there holds $\varphi_\beta^*(I, x_I) = \varphi_\beta(I, (x_{I \setminus J}, \sigma_J(x_J)))$.

Theorem 2. An autonomous facility α is regular if and only if every extension Γ^* of a regular ACLU game Γ with α admits an exact potential. The range of Γ's can be restricted to congestion games or games with structured utilities.

To ensure that the potential P attains a maximum, some additional assumptions are needed. The simplest approach would be to have P upper semicontinuous and X_N compact. A certain degree of subtlety is required, however, as was shown even in a particular case [4].

Assumption 1. The set of facilities A and each strategy set X_i are metric spaces; each mapping B_i is continuous in the Hausdorff metric on the target; for every $\alpha \in \mathrm{A}$ and $I \in \mathcal{I}_\alpha$, the function $\varphi_\alpha(I, \cdot) \colon X_I \to \mathbb{R}$ is upper semicontinuous.

For each $i \in N$ and $m \in \mathbb{N}$, we denote $X_i^m := \{x_i \in X_i \mid \#\mathrm{B}_i(x_i) = m\}$.

Assumption 2. For each $i \in N$ and $m \in \mathbb{N}$, either $X_i^m = \emptyset$ or X_i^m is a compact subset of X_i.

Assumption 3. For each $i \in N$, $X_i^m \neq \emptyset$ only for a finite number of $m \in \mathbb{N}$.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^\circ := \big\{i \in I_\alpha^+ \mid \exists O \,\big[(O \text{ is open }) \,\&\, \alpha \in O \,\&\, \forall \beta \in O\,[i \in I_\beta^+ \Rightarrow \beta = \alpha]\big]\big\}$; roughly speaking, I_α° is the set of players in whose strategy sets α is topologically isolated.

Our final assumption combines some sorts of upper semicontinuity (of φ_α in α) and monotonicity (of φ_α "in I").

Assumption 4. For every $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, and $\varepsilon > 0$, there is $\delta > 0$ such that $\varphi_\alpha(I, x_I) > \varphi_\beta(J, y_J) - \varepsilon$ whenever $\beta \in \mathrm{A} \setminus \{\alpha\}$, $J \in \mathcal{I}_\beta$, $x_I \in X_I^\alpha$, $y_J \in X_J^\beta$, $J \subseteq I \setminus I_\alpha^\circ$, and the distances between α and β in A as well as between x_J and y_J in X_J are less than δ.

If A is finite as, e.g., in a game with structured utilities or in a congestion game, then Assumption 4 holds vacuously since $I_\alpha^\circ = I_\alpha^+$, and hence no $J \in \mathcal{N}$ could satisfy the conditions.

Theorem 3. Every ACLU game satisfying Assumptions 1–4 possesses a (pure strategy) Nash equilibrium.

Dropping any one of the assumptions makes the theorem wrong.

References

1. Monderer D., Shapley L.S. Potential games. Games and Economic Behavior. 1996. V. 14. P. 124–143.
2. Rosenthal R.W. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory. 1973. V. 2. P. 65–67.
3. Kukushkin, N.S. Congestion games revisited. International Journal of Game Theory. 2007. V. 36. P. 57–83.
4. Le Breton M., Weber S. Games of social interactions with local and global externalities. Economics Letters. 2011. V. 111. P. 88–90.

Epistemic approach to Bayesian routing problem

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We highlight on the role of sharing knowledge on the users' individual conjectures on the others' selections of channels in a Bayesian routing problem. Let us consider a Bayesian extension of *KP-model*, introduced as a network game by Koutsoupias and Papadimitriou [5], and let us start to treat the simple *KP-model* consisting of one *storage* S and n *users* with which each has to use one of m *channels* to connect the storage. Each channel $l = 1, 2, \cdots, m$ has a given capacity c_l. User i intends to send/receive information with volume w_i to/from the storage S through channel l_i. The *Bayesian* KP-model is given as an extension of the KP-model equipped with a *partition information* structure.

In the seminar talk, I considered the Bayesian KP-model with partition information structure as follows. The users possess with the same prior distribution on a state-space. In addition they have private information given by a *partition information* structure i.e., a reflexive, transitive and symmetric binary relation on a state-space. Each user predicts the other players' actions as the posterior of the others' choices of channels given his/her information. I have proposed the two extended notions of equilibria, expected delay equilibrium and rational expectations equilibrium, in which the former is given as the profiles of individual conjectures such as each user maximizes his/her

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own expectations of delay and the latter is defined by the profiles of conjectures such as each user minimizes his/her own expectations of social cost respectively. Under the circumstance, In highlighting the epistemic feature I aim to give necessity condition for these equilibria as below:

Common-Knowledge Case

Theorem 1[5]. *If all users commonly know an expected delay equilibrium, then the equilibrium yields a Nash equilibrium in the based KP-model. If they commonly know a rational expectations equilibrium, then the equilibrium yields a Nash equilibrium for social cost in it.*

Common-knowledge plays essential role in the above theorem if there are more than two users. In fact, for two users case the theorem is still true without common-knowledge assumption, however for 3 users case it cannot hold without the assumption. As well known, it is actually a very strong assumption, So we would like to remove out it in our framework.

Communication Case

To the purpose we adopt the communication process introduced by Parikh and Krasucki [6] replacing common-knowledge. Let us now start that all users form a communication network. Each user sends privately his/her conjecture about the others' choices of channels to the another user according to the communication network as messages, where the message consists of information about his/her individual conjecture about the others' choices. The recipient of the message has to updates her/his private information structure by the message received. She/her has to revise her/his conjecture on the others' choices, and send the information about her/his revised conjecture to the another user according to the communication network. The users continue to communicate their private information of conjecture on the other' choices as so on. In this circumstance, we can show that

Theorem 2. *In the revision process of rational expectations equilibriums according to the communication process, the limiting conjectures yields a Nash equilibrium for social cost. For the expected delay equilibrium the same holds true also.*

Apprisals

Upper bounds for price of anarchy. By extending the notion of the price of anarchy to rational expectations equilibriums the upper bound of the price of anarchy for some typical social cost functions may be given as follows:

Conjecture. In the communication, consider the limiting expected delay equilibriums. Then the extended expected social costs for the linear social function according to the the limiting expected delay equilibrium is bounded by the ratio of the maximal capacity of the channels by the minimal one; i.e., it is lesser than or equal to $\mathrm{Max}_{i=1}^{n} c_i / \mathrm{Min}_{i=1}^{n} c_i$.

Literatures Garing et al [2] is the first paper in which Bayesian Nash equilibrium is treated. They analysis Bayesian extension of routing game specified by the type-space model of Harsanyi [3] as information structure, and they collected several results: (1) the existence and computability of pure Nash equilibrium, (2) the property of the set of fully mixes Bayesian Nash equilibria and (3) the upper bound of the price of anarchy for specific types of social function associated with Bayesian Nash equilibria.

In my work I modify their model by adopting arbitrary partition information structure following Aumann [1] instead of the type-space model. The merit of adopting information partition structure lies not only in getting the close connection to computational logic but also in increasing the range of its applications in various fields.

It ends well by remarking on the assumption in the model. I have treated the volumes in the Bayesian KP-model as *indivisible* goods, but we should treat it as *divisible* ones when KP-model is considered as a model of cloud computing system, because the volumes will be given as the volumes of information, which is considered as divisible. Furthermore, it will have to arise several interesting problems to investigate in future agendas. Among others the most important is to study the several core notions appeared in our framework of Bayesian game.

References

1. Aumann R.J. Agreeing to disagree// Annals of Statistics. 1976. V. 4. P. 1236–1239.
2. Garing M., Monien B., Tiemann K. Selfish routing with incomplete information// Theory of Computing Systems. 2008. V. 42. P. 91–130.

3. Harsanyi J.C. Games with incomplete information played by Bayesian players, I, II, III// Management Science. 1967. V. 14. P. 159–182, 320–332, 468–502.
4. Koutsoupias E., Papadimitriou C.H. Worst-case equilibria// In: Meinel C. and Tison S.(eds). Proceedings of the 16th International Symposium of Theoretical Aspect of Computer Science. Lecture Notes in Computer Science. 1999. V. 1563. P. 404–413.
5. Matsuhisai T. Selfish routing with common-knowledge. Working Paper. 2015.
6. Parikh R., Krasucki P. Communication, consensus and knowledge// Journal of Economic Theory. 1990 V. 52. P. 78–89.

Minimax estimation of the parameter of the negative binomial distribution*

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Let's consider the minimax estimation problem of the parameter θ of the negative binomial distribution (NBD) $f(t|\theta,r)=\theta^r(1-\theta)^t(r)_t/t!$, $t=0,1,...$, where $(r)_t=r(r+1)\cdots(r+t-1),\ t\geqslant 1, (r)_0=1$. The parameter $r>0$ is assumed to be known. We use the quadratic loss function $L(\theta,d)=(\theta-d)^2$. For the geometric distribution $(r=1)$ a statistical game was solved by G.N. Dyubin in [1]. Here a similar solution is obtained for $r\in(0,1)$. If $r>1$, a numerical method is specified for finding a minimax estimator. When $r>2$, the estimate, which minimizes the maximum risk among linear estimates of the form $c_1\delta_0+c_2$, where δ_0 is an unbiased estimator, is constructed.

Problem. A statistician observes a value t of the random variable T having NBD $f(t|\theta,r)$. A decision function $\delta:\mathbb{Z}_+\to[0,1]$ is a strategy of the statistician belonging to the set Δ of all such strategies. After the substitution of the strategy δ in the loss function L and subsequent averaging over $f(t|\theta,r)$, one obtains the risk function

$$R(\theta,\delta)=\mathrm{E}[L(\theta,\delta(T))|\theta]=\theta^r\sum_{t=0}^{\infty}\frac{(r)_t}{t!}(1-\theta)^t(\theta-\delta(t))^2.$$

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In the statistical game $G = \langle [0,1], \Delta, R(\theta,\delta)\rangle$ the first player (nature) maximizes the risk function R, and the second player (statistician) minimizes it. It's assumed that the nature may use mixed strategies $\xi \in \Xi$.

Solution of the game for $r \in (0,1)$. Let $\theta_0 \in (0,1)$ be a root of the equation $\theta(2\theta^r/2 + r + 2) = r$ and $\lambda_0 = (r - (r+2)\theta_0)/(2\theta_0^{r+1} + r - (r+2)\theta_0)$. We denote by I_θ the indicator of point θ.

Proposition 1. *If* $r \in (0,1)$, *then* $\xi^* = \lambda_0 I_{\theta_0} + (1-\lambda_0)I_1$ *and* $\delta^*(0) = (1+2/r)\theta_0$, $\delta^*(t) = \theta_0$, $t = 1, 2, \ldots$ *are the optimal strategies of the players, and* $v = (1-\delta^*(0))^2$ *is the value of the game* G.

Minimax linear estimator. A strategy of interest is the linear estimate δ^l, which minimizes the maximum risk on $\Delta^l = \{c_1\delta_0 + c_2 | c_1, c_2 \in [0,1]\}$ (see [2]). For any $\delta = c_1\delta_0 + c_2 \in \Delta^l$ risk function can be written as $F(\theta, c_1, c_2) \overset{def}{=} R(\theta,\delta) = (\theta(1-c_1) - c_2)^2 + c_1^2(\theta^r h(\theta) - \theta^2)$, where

$$h(\theta) = \sum_{t=0}^{\infty} \frac{(r-1)_t^2}{t!(r)_t}(1-\theta)^t = \frac{r-1}{(1-\theta)^{r-1}} \int_0^{(1-\theta)/\theta} \frac{z^{r-2}}{1+z}dz$$

is a generalized hypergeometric function. To find the strategy $\delta^l = c_1^l\delta_0 + c_2^l$, we solve the game $G^l = \langle [0,1], [0,1]^2, F(\theta, c_1, c_2)\rangle$. Consider the following system of equations for the variables θ, c_1, c_2 :

$$F_{c_1}(\theta, c_1, c_2) = 0,\ F_\theta(\theta, c_1, c_2) = 0,\ F(\theta, c_1, c_2) = F(0, c_1, c_2). \qquad (1)$$

Lemma. *For* $r > 2$ *the system of equations* (1) *has a unique solution.*

Proposition 2. *For* $r > 2$ *let* (θ^l, c_1^l, c_2^l) *be the solution of* (1). *Denote* $\lambda^l = c_2^l/(\theta^l(1-c_1^l))$. *Then* $\xi^l = \lambda^l I_{\theta^l} + (1-\lambda^l)I_0$ *and* $\delta^l = c_1^l\delta_0 + c_2^l$ *are optimal strategies for players and* $v^l = (c_2^l)^2$ *is the value of the game* G^l.

An approximate solution of the game. Let's consider $r > 1$. For integer $N \geqslant 1$ we define a truncated strategy $\delta_N = (\delta(0), \delta(1), \ldots, \delta(N)) \in [0,1]^{N+1}$ and the corresponding payoff function

$$R_N(\theta, \delta_N) = \theta^r \sum_{t=0}^{N} \frac{(r)t_t}{!}(1-\theta)^t(\theta - \delta(t))^2$$

of the game $G_N = \langle [0,1], [0,1]^{N+1}, R_N(\theta, \delta_N)\rangle$. The function R_N is convex in δ_N. Therefore in the game G_N the nature may use mixed strategies of the form $\xi=\sum_{i=1}^m a_i I_{\theta_i}$, where

$$\sum_{i=1}^m a_i=1,\ a_i\geqslant 0,\ i=1,...,m,\ 0\leqslant\theta_1\leqslant\theta_2\leqslant...\leqslant\theta_m\leqslant 1,\ m\leqslant N+2.$$

The set of all such strategies is denoted by Ξ^m. For each strategy $\xi\in\Xi^m$ the corresponding Bayesian strategy $\delta_N^\xi=(\mathrm{E}\,[\Theta|t],t=0,1,...,N)$ minimizes $R_N(\xi,\delta_N)$ for $\delta_N\in[0,1]^{N+1}$, where the expectations $\mathrm{E}\,[\Theta|t]$ are taken over the posterior distribution

$$\xi|t=\sum_{i=1}^m a_i\theta_i^r(1-\theta_i)^t I_{\theta_i}\Big/\sum_{j=1}^m a_j\theta_j^r(1-\theta_j)^t.$$

To solve approximately the game G_N, we fix the accuracy $\varepsilon_1>0$ and choose a value of $m<N+2$. We have

$$\underline{v}_N=\max_{\xi\in\Xi^m}R(\xi,\delta_N^\xi)=R(\xi^*,\delta_N^{\xi^*})\leqslant v_N\leqslant\bar{v}_N=\max_{\theta\in[0,1]}R(\theta,\delta_N^{\xi^*}).$$

If inequality $\bar{v}_N-\underline{v}_N\leqslant\varepsilon_1$ is not satisfied, we increase m and repeat the calculations to achieve the accuracy ε_1. Note that for a given ε_1 a minimal required m grows with r. The following table shows the minimal m, which ensures the accuracy $\varepsilon_1=10^{-8}$:

r	2	3	4	5	6	7	8	9	10
m	10	12	14	21	25	26	28	30	32

Using the found strategy $\delta_N^{\xi^*}$ let's define a strategy δ^* in the original game G :

$$\delta^*(t)=\begin{cases}\delta_N^{\xi^*}(t), & 0\leqslant t\leqslant N,\\ \delta_N^{\xi^*}(N), & t>N.\end{cases}$$

The strategy δ^* realizes $\min\limits_{\delta\in\Delta}\max\limits_{0\leqslant\theta\leqslant 1}R(\theta,\delta^*)$ with $\varepsilon>0$. To get ε, we find an upper bound for a «tail» of series $R(\theta,\delta^*)$:

$$\sum_{t>N}\frac{(r)_t}{t!}\theta^r(1-\theta)^t(\theta-\delta^*(t))^2\leqslant\max_{0\leqslant\theta\leqslant 1}\theta^2\Big(1-\sum_{t=0}^N\frac{(r)_t}{t!}\theta^r(1-\theta)^t\Big)\stackrel{def}{=}\varepsilon_2(N).$$

Now we can take $\varepsilon=\varepsilon_1+\varepsilon_2(N)$. It should be noted that $\varepsilon_2(N)$ decreases slowly with growth of N. For example, if $r=4$ $\varepsilon_2(200)\approx 0.00017$, and $\varepsilon_2(1000)\approx 0.000007$. At large N the solution of the game

G_N requires a significant amount of computations. It is possible to reduce $\varepsilon_2(N)$ with the following method. The statistician suggests that $\theta \in [\underline{\theta}, 1]$, where $\underline{\theta} > 0$ is a lower bound of the parameter θ. Then in games G and G_N one needs to change the interval $[0,1]$ to $[\underline{\theta}, 1]$, and calculate $\varepsilon_2(N)$ as maximum on $[\underline{\theta}, 1]$. For example, if $r = 4$ and $\underline{\theta} = 0.1$ the improved value of $\varepsilon_2(200)$ equals 10^{-8}. So, the value of the game v is 0.01943937 with $\varepsilon = \varepsilon_1 + \varepsilon_2(200) = 2 \cdot 10^{-8}$.

References

1. Dyubin G.N. The statistical game of the estimation of geometric distribution parameter// Game-theoretical questions. Leningrad: Nauka, 1978. P. 124–125.
2. Ferguson T.S, Kuo L. Minimax estimation of a variance// Annals of the Institute of Statistical Mathematics. 1994. V. 46, №2. P. 295–308.

Games with polynomials*

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The following class of antagonistic games is considered [1]: the polynomial is given

$$f : f(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m. \tag{1}$$

Two players change alternately one coefficients a_i by any real number, but each coefficient is used only one time.

Payoff function of the first player (player who makes the first move) is determined by one of two following ways:

a) $H_1(s_1, s_2)$ is the amount of different real roots of the polynomial f;

b) $H_2(s_1, s_2) = -H_1(s_1, s_2)$

(s_1 — is the strategy of the first player, s_2 — is the strategy of the second player).

It means that in the case a) the first player strives to that the polynomial f had most of all different real roots and in the case b) the first player strives to that the polynomial f had least of all different real roots. The aim for second player is opposite.

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Let $v_j(m, a_m)$ is value of the game with payoff function H_j of first player and with m-th degree polynomial (1), where a_m is constant term.

Theorem. That is true

1. $v_1(2n+1, a_{2n+1}) = v_1(2n+1, -1) = 1$ for all $n > 1$;
2. $v_1(2n, a_{2n}) = 2$ for all $n \geq 1$;
3. $v_1(2n, -1) = 4$ for all $n \geq 3$, $v_1(4, -1) = 2$;
4. $v_2(2n, a_{2n}) = 4$ for all $n \geq 4$, $v_2(4, a_4) = 2$;
5. $v_2(2n, -1) = 2$ for all $n \geq 2$;
6. $3 \leq v_2(2n+1, -1) \leq 5$ for all $n \geq 2$, $v_2(3, -1) = 3$.

References

1. Petrov N.N. About the one polynomial game// Matematicheskaya Teoriya Igr i Ee Prilozheniya. 2012. V. 5(3). P. 58–71 (in Russian).

Multistage bidding model with elements of bargaining: extension for a countable state space*

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We consider a simplified model of a financial market with two players bidding for one unit of a risky asset for $n \leq \infty$ consecutive stages. Player 1 (an insider) is informed about the liquidation price s^0 of the asset while Player 2 knows only its probability distribution p. At each stage players place integral bids. The higher bid wins, and an asset is transacted to the winning player. Each player aims to maximize the value of her final portfolio.

A model where the price s^0 has only two possible values $\{0, m\}$ is considered in [1]. It is reduced to a zero-sum game $G_n(p)$ with incomplete information on one side as in [2]. In this model uninformed Player 2 uses the history of Player 1's moves to update posterior probabilities over the liquidation price. Thus, Player 1 should find a strategy controlling posterior probabilities in such a way that allows her to use the private information without revealing too much of it to Player 2. In [3] the model is extended so that the liquidation price can take any value $s \in S = \mathbb{Z}_+$ according to a probability distribution $p = (p_s, s \in S)$. It is shown that when $\mathbb{D}p$ is finite, a game $G_\infty(p)$ is properly defined. For this game the value and optimal players strategies are found.

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In both [1] and [3] the transaction price equals to the highest bid. Instead we can consider a transaction rule proposed in [4], and define a transaction price equal to a convex combination of proposed bids with a coefficient $\beta \in [0,1]$. A model with such transaction rule and two possible values of the liquidation price is studied in [5]. Here those results are further extended for the case of a countable state space.

The model is defined as follows. At stage 0 a chance move chooses a state of nature $s^0 \in S$ according to the distribution p. At each stage $t = \overline{1,n}$ players make bids $i_t \in I, j_t \in J$ where $I = J = \mathbb{Z}_+$. A stage payoff in state s equals to

$$a^s(i_t, j_t) = \begin{cases} (1-\beta)i_t + \beta j_t - s, & i_t < j_t, \\ 0, & i_t = j_t, \\ s - \beta i_t - (1-\beta)j_t, & i_t > j_t. \end{cases}$$

Player 1's strategy is a sequence of actions $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_t : \ S \times I^{t-1} \to \Delta(I)$ is a mapping to the set of probability distributions $\Delta(I)$ over I. So, at each stage of the game Player 1 randomizes his bids depending on the history before stage t and the state s. Player 2's strategy is defined as a sequence of actions $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_t : \ J^{t-1} \to \Delta(J)$. The payoff in this zero-sum game $G_n(p)$ is defined as

$$K_n(p, \sigma, \tau) = \mathbb{E}_{(p,\sigma,\tau)} \sum_{t=1}^{n} a^s(i_t, j_t).$$

Let's denote distribution sets $\Theta(x) = \{p' \in \Delta(S) : \mathbb{E}p' = x\}$ and $\Lambda(x, y) = \{p' \in \Delta(S) : x < \mathbb{E}p' \leq y\}$. Similar to [3], it can be shown that for $p \in \Lambda(k-1+\beta, k+\beta)$ a pure strategy τ^k defined as

$$\tau_1^k = k, \quad \tau_t^k(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1}, & i_{t-1} < j_{t-1}, \\ j_{t-1}, & i_{t-1} = j_{t-1}, \\ j_{t-1}, & i_{t-1} > j_{t-1}, \end{cases}$$

guarantees to Player 2 a payoff not greater than $H_\infty(p)$ in game $G_n(p)$. Function $H_\infty(p)$ is piecewise linear with breakpoints at $\Theta(k+\beta)$ and domains of linearity $\Lambda(k-1+\beta, k+\beta)$. For distribution p such that $\mathbb{E}p = k-1+\beta+\xi, \ \xi \in [0,1)$, it equals to

$$H_\infty(p) = (\mathbb{D}p + \beta(1-\beta) - \xi(1-\xi))/2.$$

Since $\mathbb{D}p$ is assumed finite, the value $H_\infty(p)$ is finite as well. Hence an infinitely long game $G_\infty(p)$ can be considered.

Let's denote $L_\infty(p)$ a guaranteed payoff to Player 1 in game $G_\infty(p)$, and $p^x(l,r) \in \Theta(x)$ a probability distribution taking only values l and r. It can be shown that Player 1 can guarantee herself for $p = \lambda p_1 + (1-\lambda)p_2$ a payoff of at least $\lambda L_\infty(p_1) + (1-\lambda)L_\infty(p_2)$. Since every distribution p can be represented as a convex combination of some $p^x(l,r)$, proving that $H_\infty(p) = L_\infty(p)$ requires an explicit proof only for $p = p^{k+\beta}(l,r)$.

Let's denote $q = (q_i, i \in I)$ a marginal distribution of Player 1's first bid and $p^i = (p^{s|i}, s \in S)$ a posterior distribution over the liquidation price given a bid i was made. Let's also denote σ_i^s a component of Player 1's stage action, i.e. a probability of making a bid i in state s. Then from the Bayes rule $\sigma_i^s = p^{s|i} q_i / p_s$. Thus in order to define a stage action, it is suffice to specify q and $(p^i, i \in I)$.

An optimal strategy for $p^x(0,m)$ as described in [5] can be adjusted to $p^{k+\beta}(l,r)$ in the following way. For $p = p^l(l,r)$ and $p = p^r(l,r)$ Player 1 uses bids l and r respectively with probability 1 at the first stage of the game. For $p \in \left\{p^k(l,r), p^{k+\beta}(l,r)\right\}$ she uses a stage action with parameters

$$p^k(l,r) : q_k = \beta, q_{k+1} = 1-\beta, p^k = p^{k-1+\beta}(l,r), p^{k+1} = p^{k+\beta}(l,r),$$
$$p^{k+\beta}(l,r) : q_k = 1-\beta, q_{k+1} = \beta, p^k = p^k(l,r), p^{k+1} = p^{k+1}(l,r).$$

Applied recursively for respective posterior probabilities at subsequent stages this strategy guarantees to Player 1 a payoff at least

$$L_\infty\left(p^{k+\beta}(l,r)\right) = \big((r-k-\beta)(k-l+\beta) + \beta(1-\beta)\big)/2.$$

This coincides with the value of $H_\infty\left(p^{k+\beta}(l,r)\right)$. Thus the game $G_\infty(p)$ has a value $V_\infty(p) = H_\infty(p)$, and strategies described above are optimal.

It must be noted that Player 2's strategy is surprisingly robust in regard to changes in the payoff function. At the same time Player 1's strategy becomes more complex. For initial $p \in \Theta(k)$ posterior probabilities in [3] form a symmetric random walk, i.e. posterior p' will be either in $\Theta(k-1)$ or $\Theta(k+1)$ with equal to 1/2 probabilities. This is no longer true when $\beta \in (0,1)$. The strategy described above essentially differs from that in [3], e.g. it doesn't collapse to that of [3] when $\beta \to 1$.

References

1. Domansky V. Repeated games with asymmetric information and random price fluctuations at finance markets // International Journal of Game Theory. 2007. V. 36, № 2. P. 241–257.

2. Aumann R.J., Maschler M.B. Repeated Games with Incomplete Information. Cambridge, Massachusetts: The MIT Press, 1995.
3. Domansky V.C., Kreps V.L. Game theoretic bidding model: strategic aspects of price formation at stock markets // The Journal of the New Economic Association. 2011. V. 11. P. 39–62.
4. Chatterjee K., Samuelson W. Bargaining under incomplete information // Operations Research. 1983. V. 31, №5. P. 835–851.
5. P'yanykh A.I. A Multistage exchange trading model with asymmetric information and elements of bargaining // Moscow University Computational Mathematics and Cybernetics. 2016. V. 40, № 1. P. 35–40.

Equilibria in dynamic multicriteria games*

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Mathematical models involving more than one objective seem more adherent to the real problems. Often players have more that one goal and they can be not comparable. These situations are typical for game-theoretic models in economy and ecology. Hence, multicriteria game approach helps to make decisions in multi-objective problems.

Shapley [4] introduced the concept of multicriteria games that are games with vector payoffs, and gave a generalization of classical Nash equilibrium to Pareto equilibrium for such games. In recent years, many authors have studied the game problem with vector payoffs. Some concepts have been suggested to solve multicriteria games: in [5] it was presented the notion of ideal Nash equilibrium, [1] connected multicriteria game with potential game and [2] suggested E-equilibrium concept.

Traditionally, equilibrium analysis in multicriteria problems bases on the static or steady-state variant. For dynamic multicriteria games proposed equilibrium concepts do not assist in evaluating players' behavior. Presented work is dedicated to linking multicriteria games with dynamic games. The new approach to construct the equilibrium in dynamic game with many objectives is proposed.

We consider a bicriteria dynamic game with two participants in discrete time. Players exploit the common resource and both wish to

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optimize two different criteria. The state dynamics is in the form

$$x_{t+1} = f(x_t, u_{1t}, u_{2t})\,, \quad x_0 = x\,, \tag{1}$$

where $x_t \geq 0$ is the resource size at step t, $u_{it} \in U_i$ indicates the strategy of player i, $i = 1, 2$.

The payoff functions of the players over infinite time horizon are defined by

$$J_1 = \begin{pmatrix} J_1^1 = \sum\limits_{t=0}^{\infty} \delta^t g_1^1(u_{1t}, u_{2t}) \\ J_1^2 = \sum\limits_{t=0}^{\infty} \delta^t g_1^2(u_{1t}, u_{2t}) \end{pmatrix}, \; J_2 = \begin{pmatrix} J_2^1 = \sum\limits_{t=0}^{\infty} \delta^t g_2^1(u_{1t}, u_{2t}) \\ J_2^2 = \sum\limits_{t=0}^{\infty} \delta^t g_2^2(u_{1t}, u_{2t}) \end{pmatrix}, \tag{2}$$

where $g_i^j(u_{1t}, u_{2t}) \geq 0$ gives the instantaneous utility, $i, j = 1, 2$, $\delta \in (0, 1)$ means the common discount factor.

In the present work we design the equilibrium in multicritetia game using the Nash bargaining solution. Therefore, we begin with construction of guaranteed payoffs which play the role of the status quo points.

There are three possible concepts to determine the guaranteed payoffs G_1^1, G_1^2, G_2^1, G_2^2.

In the first one four guaranteed payoff points are obtained as the solutions of zero-sum games. In particular, the first guaranteed payoff point is a solution of zero-sum game where player 1 wishes to maximize her first criterion and player 2 wants to minimize it. Other points are obtained by analogy.

The second approach can be applied when the players' objectives are comparable. Consequently, the guaranteed payoff points for player 1 are obtained as the solution of zero-sum game where she wants to maximize the sum of her criteria and player 2 wishes to minimize it. And, by analogy, for player 2.

In the third approach the guaranteed payoff points are constructed as the Nash equilibrium with the first and the second criteria of both players, respectively.

To construct multicriteria payoff functions we adopt the Nash products. The role of the status quo points belongs to the guaranteed payoffs of the players:

$$H_1(u_{1t}, u_{2t}) = (J_1^1(u_{1t}, u_{2t}) - G_1^1)(J_1^2(u_{1t}, u_{2t}) - G_1^2)\,, \tag{3}$$

$$H_2(u_{1t}, u_{2t}) = (J_2^1(u_{1t}, u_{2t}) - G_2^1)(J_2^2(u_{1t}, u_{2t}) - G_2^2)\,. \tag{4}$$

Next definition presents the suggested solution concept.

Definition. *Strategy profile* (u_{1t}^*, u_{2t}^*) *is called multicriteria Nash equilibrium of the problem (1)–(2) if*

$$H_1(u_{1t}^*, u_{2t}^*) \geq H_1(u_{1t}, u_{2t}^*) \ \ \forall u_{1t} \in U_1 \,, \tag{5}$$

$$H_2(u_{1t}^*, u_{2t}^*) \geq H_2(u_{1t}^*, u_{2t}) \ \ \forall u_{2t} \in U_2 \,. \tag{6}$$

Just like in classical Nash equilibrium approach it is not profitable for both players to deviate from equilibrium strategies. But under presented equilibrium concept players maximize the product of the differences between optimal and guaranteed payoffs (3)–(4).

A dynamic multicriteria model related with the bioresource management problem (fish catching) is investigated to show how suggested concept works.

References

1. Patrone F., Pusillo L. and Tijs S.H. Multicriteria games and potentials // Top. 2007. V. 15. P. 138–145.
2. Pusillo L., Tijs S. E-equilibria for multicriteria games // Annals of ISDG. 2013. V. 12. P. 217–228.
3. Rettieva A.N.A discrete-time bioresource management problem with asymmetric players // Automation and Remote Control. 2014. V. 75(9). P. 1665–1676.
4. Shapley L.S. Equilibrium points in games with vector payoffs // Naval Research Logistic Quarterly. 1959. V. 6. P. 57–61.
5. Voorneveld M., Grahn S. and Dufwenberg M. Ideal equilibria in noncooperative multicriteria games // Mathematical Methods of Operations Research. 2000. V. 52. P. 65–77.

Analysis of political processes and corruption

The phenomena of soft power and double standards in mathematical model of cross-cultural interaction

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The article presents the results of mathematical modeling of cross-cultural interaction by the competition equations. Study of the model finds the possibility of a paradox situation, when one of the cultures positively treats the other, though this other one is actually quite harmful to it. Conversely, in some cases, quite a harmless culture can be treated as very negative one.

Double standards are characterized by different application of the principles, laws, rules, estimates to the same actions of various subjects, depending on the degree of loyalty of these subjects to the estimator or other reasons of benefit for him. As for the soft power - this term was for the first time introduced in 1990 by Joseph Nye of Harvard University [3], but something similar can be found also in works of Antonio Gramsci and even in the ancient time - in Laozi's Tao Te Ching. It is possible to say, that the cultural values capable to induce others to want what is wanted by you, are the cornerstone of the concept of soft power.

In the work [1] an interaction of two cultures was modelled by

A. Lotka and V. Volterra competition equations.

$$\frac{dN}{dt} = \alpha N\left(1 - \frac{N}{N^*} - m\frac{M}{M^*}\right), \quad \frac{dM}{dt} = \beta M\left(1 - \frac{M}{M^*} - n\frac{N}{N^*}\right). \quad (1)$$

Here we treat a culture on its household level - as a certain method of behavior, i.e. as a set of standard reactions to standard requests of the environment. In our elementary model (1) we select from this set only two factors: an attitude to compatriots and an attitude to strangers.

In the same work [1] it was shown that the behavior of this system of equations first of all depends on coefficients of intolerance n and m. It would also be possible to call these coefficients by double standards factors – they show in how many times the competition in the culture more or less than its competition with the foreign one.

We shall distinguish the following ranges of these double standard coefficients:

- Supertolerance, if $-\infty < n, m < 0$.
- Tolerance, if $0 \leq n, m < 1$.
- Treatment without prejudices and preferences, when n and m equals to one (no double standards).
- Intolerance, when $0 < n, m < \infty$.

It occures [1], that if the double standard coefficients are lesser than one (tolerance), the cultures are friendly - they can exist together. If the double standard coefficient of a culture is greater than one (intolerance) - this culture constitutes a real danger to another - may force it out from the system.

Besides, the capability of social systems to change the behavior on short times in response to current situation, turns the dynamic system (1) into a position differential game [2], where the double standard coefficients n and m become the controls of players.

That is why double standards are so popular in the interstate relations. Nevertheless, in the work [1] it is shown that if the rivals are equally strong, uncontrolled increase in mutual intolerance (use of double standards), becomes equally dangerous to both players. In this case there are other interesting strategies of the game [2].

Now let us look at a situation, for example, from the position of culture N representative. First, the value $\frac{N}{N^*}$ is well-known to him, because this value is a way of attitude to compatriots in the culture N - a way of good behavior which is taught since the childhood. Secondly, the value $m\frac{M}{M^*}$ is also known - it is a cometition pressure of the culture

M, which the representatives of the culture N directly observe, because they are under this pressure. Most likely, these values are not identical $\frac{N}{N^*} \neq m\frac{M}{M^*}$ - because the cultures are really different.

Further, it is quite natural to assume that if $\frac{N}{N^*} > m\frac{M}{M^*}$, then it is pleasant to the representative of the culture N - usually it is pleasant to anybody, when the pressure upon him weakens. Perhaps, he assesses this situation approximately so: "Ah, what darlings, these well-mannered people of M - not that my rough compatriots!"On the contrary, if $\frac{N}{N^*} < m\frac{M}{M^*}$, then representative of N does not like this fact - very few people like the pressure bigger than usual. Most likely he will think: "Well and how savage are these M! It is quite impossible to live nearby them! They are not able to behave at all!"

Actually, both as the first, either the second estimate can be deeply wrong - in the system (1) nothing depends upon the ratio between the values $\frac{N}{N^*}$ and $m\frac{M}{M^*}$, as well as from the ratio between $\frac{M}{M^*}$ and $n\frac{N}{N^*}$. The behavior of the system (1) depends only upon the double standard factors n and m [1].

For example, if $\frac{N}{N^*} >> m\frac{M}{M^*}$, but at the same time $m > 1$ - the situation can be dangerous for the culture N, it can disappear completely over a time, because of the neighborhood with "lovely and well-mannered" people, especially if it puts $n \leq 1$, having been under illusion of the first inequality.

On the contrary, if $\frac{N}{N^*} < m\frac{M}{M^*}$ and even $\frac{N}{N^*} << m\frac{M}{M^*}$, but $m < 1$ - there is no danger for the culture N to disappear near the culture M. Moreover, if $n > 1$ - the culture N forces out the rival trough a time.

However, if the system (1) becomes a differential game, the double standard factors n and m are not observed directly. For the representative of the culture N to define m, is necessary to compare given him in feelings $m\frac{M}{M^*}$ with $\frac{M}{M^*}$, but the last value, as a rule is unknown to him: studying of foreign cultures is a destiny of rather narrow circle of specialists.

Thus,this elementary model learns us that it is incorrect to measure one culture by the gauge of another – such a measurement is not valid. The only true yardstick for the culture is this culture itself, i.e. the competitive pressure of a foreign culture is to be compared with its own internal competition, but by no means with the internal competition of the native culture.

At the author's subjective view, this paradox illustrates why our cutting through a "window to Europe" during the last 300 years is not too successful. The Slavs once lived in Europe, but little from them remained. At the same time, under the Horde Yoke we survived, and under the Ottoman Empire the southern Slavs did, though very unpleasant memoirs about these History periods remained in the folklore of survivors.

References

1. Brodsky Yu.I. Tolerance, Intolerance, Identity: simple math models of cultures' interaction. Saarbrucken: LAP LAMBERT Academic Publishing, 2011.
2. Brodsky Yu.I. Cross-Cultural Interaction as a Positional Differential Game // Facing an Unequal World: Challenges for Russian Sociology, Editor-in-Chief V. Mansurov, Moscow-Yokohama, 2014. P. 313-316.
3. Nye J. S. Soft Power: The Means to Success in World Politics. N.Y.: Public Affairs, 2004.

Markets and auctions: analysis and design

Nash-2 equilibrium: how farsighted behavior affects stable outcomes*

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In a bounded rationality framework, modeling iterated strategic thinking process becomes more and more complicated as the number of participants increases. Most papers have been devoted to analysis of 2-person games with non-trivial agents expectations on the opponent's reaction and various depths of such mutual predictions. The common approach to n-person games with $n > 2$ is to introduce cognitive hierarchy of players (see survey [1]). This requires certain knowledge of the opponent's calculation abilities. However in many real-life situations, players might face an uncertainty how sophisticated their competitors are. In particular, the opponents' levels of rationality evolve in the course of the game [2]. In this case an accurate prediction of response even at one step ahead seems to be unreasonable.

The paper [5] introduces an equilibrium concept, so-called Nash-2 equilibrium, in 2-player games with the following idea. A player supposes that any profitable response of the opponent might follows on her deviation and rejects such own improvement that may lead to

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poorer situation after some opponent's reaction. [5] provides a complete characterization of Nash-2 equilibrium, resolves existence problem, discusses the relation with equilibrium in threats and counter-threats, equilibrium in secure strategies, sequentially stable set, equilibrium in double best responses, and contains convincing examples why such equilibria can sometimes explain tacit collusion and more effective outcomes than Nash equilibrium.

In this work I extend the definition on Nash-2 equilibrium to n-person non-cooperative games. The underlying intuition is based on spatial economics notion of direct and indirect competitors [3]. In a game with large number of players it is natural to assume that each player divides her opponents into direct competitors whose reaction she worries about and tries to predict, and indirect competitors whose strategy is believed to be fixed as in Nash equilibrium concept. Such a selective farsightness looks more plausible than total ignorance of reactions or perfect prediction of future behavior of all other competitors.

Consider an n-person non-cooperative game in the normal form $G = (i \in I = \{1, \ldots, n\}; \quad s_i \in S_i; \quad u_i : S_1 \times \ldots \times S_n \to \mathbb{R})$, where s_i, S_i and u_i are the strategy, the set of all available strategies and the payoff function, respectively, of player i, $i = 1, \ldots, n$.

Let us define the **reflection network** g by the following rule. Nodes are players i in I. A *directed* link $g_{ij} = 1$ from player i to j means that player i accounts profitable responses of player j in her reasoning. $g_{ij} = 0$, otherwise. Denote by $N_i(g)$ the set of neighbours j of player i in the graph g, such that $g_{ij} = 1$.

Definition 1. A deviation s'_i of player i at profile $s = (s_i, s_{-i})$ is **secure** if for any subset $J \subseteq N_i(g)$ and any profitable deviation s'_j of every player $j \in J$ at intermediate profile (s'_i, s_{-i}) even in case of simultaneous deviations of all players from J player i is not worse off, i.e. $u_i(s'_i, s'_J, s_{-iJ}) \geq u_i(s)$.

We maintain a non-cooperative framework and assume that if player i has several out-neighbors they will not coordinate their actions. In this paper the reflection network is fixed exogenously before the game starts, it is not a result of strategic choice.

Note that if $N_i(g) = \emptyset$ then player i does not worry about any possible reactions, and so *every* her deviation is secure by definition. We will call this situation *fully myopic behavior*.

Definition 2. A strategy profile is **a Nash-2 equilibrium** if no player has a profitable and secure deviation.

Every Nash equilibrium is also a Nash-2 equilibrium irrespective of the architecture of the reflection network. Moreover, in the case of empty reflection network they are coincide by definition. In general, non-trivial reflection network significantly influences equilibrium outcomes. A striking example is **prisoner's dilemma**.

Consider the model of n-player prisoner's dilemma from [4]. Each player has two possible strategies: to cooperate with the community or to defect. The utility function is

$$u_i = \begin{cases} bA/n - c, & \text{if player } i \text{ cooperates,} \\ bA/n, & \text{if player } i \text{ defects,} \end{cases}$$

where A is a number of cooperators in the game, each of them brings profit b to the society, but pays the cost c. The total profit is equally divided to all n players irrespective of their real contribution. Unilateral defection is preferred for each individual $c > \frac{b}{n}$; overall cooperation is more preferred for each player than common defection $b > c > 0$.

Though under Nash rationality, cooperation is unlikely to emerge, even in evolutionary game setting, considering a non-empty reflection network yields cooperation. The number of cooperators depends both on the architecture of network and the relation between b and c. Assume that A players cooperate and any cooperator i reflects about n_i other cooperators. Such a situation is a Nash-2 equilibrium if and only if

$$n_i > n^* = \frac{cn}{b} - 1, \quad A > \frac{cn}{b}.$$

This means that a player reflecting about relatively small number of agents never cooperates. Therefore, in Nash-2 equilibrium *any subset of players with sufficient number of "links"with the other cooperators are able to maintain cooperation* while all other defect if the number of cooperators is enough to provide positive profits for cooperators. When these profits are very small the cooperation requires the complete reflection network among cooperators. Hence, for supporting cooperative behavior it is important not only to provide a balance between the value of individual return and the cooperation cost, but also to *ensure close contacts between cooperators.*

Further examples will include analysis of oligopoly with different structures of reflection networks. The connection with spatial models will be highlighted. Common patterns of reflection networks will be identified.

References

1. Crawford V., Costa-Gomes M., Iriberri N. Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and

Applications // Journal of Economic Literature. 2013. Vol. 51, No. 1. P. 5-62.
2. Frey S., Goldstone R. Flocking in the depths of strategic iterated reasoning // arXiv preprint arXiv:1506.05410. 2015.
3. Gabszewicz J.J., Thisse J.-F. Spatial competition and the location of firms. In: Location Theory (Fundamentals of Pure and Applied Economics, 5). 1986. P. 1-71.
4. Rezaei G., Kirley M., Pfau J. Evolving cooperation in the n-player prisoner's dilemma: A social network model // Artificial Life: Borrowing from Biology. Springer Berlin Heidelberg, 2009. P. 43-52.
5. Sandomirskaia M. Rational decision making under uncertainty of reaction: Nash-2 equilibrium concept // Working paper WP7/2016/01. Series WP7 "Mathematical methods for decision making in economics, business and politics". 2016. P.1-40.

Optimization of energetic markets' transport infrastructure*

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Markets of natural gas, oil and electricity play an important role in economies of many countries. Every such market includes its own transmission system. Consumers and producers are located at different nodes, and transmission capacities of the lines between the local markets are limited. The share of transport costs in the final price of the resource is typically substancial, the problem of transmission system optimization is of practical interest. Paper [1] determines the optimal transmission capacity for a two-node market. The present study considers a general problem of social welfare optimization with account of production costs, consumers' utilities and costs of trasmission capasities' increments. The complexity of the problem concerns with substancial fixed costs related to expansions of transmitting lines. If the set of expanded lines were given the problem would be convex and could be solved by standard methods. However, under a big number of lines the efficient search of the set requires special tools. In general the problem of transport system optimization is NP-hard (see [2]). Below we determine conditions for

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submodularity and for supermodularity of the social welfare function on the set of transmitting lines. These properties provide a possibility to apply the known efficient optimization methods (see[3],[4]).
We consider a homogeneous good market consisting of several local markets and a network transmission system. Let N denote the set of nodes and $L \subseteq N \times N$ be the set of edges. Every node $i \in N$ corresponds to a local perfectly competitive market. Demand function $D_i(p)$ and supply function $S_i(p)$ characterize respectively consumers and producers in the market and meet standard conditions. The demand function relates to the consumption utility function: $U_i(q) = \int_0^q D_i^{-1}(v)\,dv$. The supply function $S_i(p)$ determines the optimal production volume at the node i : $S_i(p) = Arg\ \max_v (pv - c_i(v))$, where $c_i(v)$ is the minimal production cost of volume v at node i. The total profit of producers at node i under price $\overline{p}$ is $Pr_i(\overline{p}) = \int_0^{\overline{p}} S_i(p)dp$. For any $(i,j) \in L$, the line is characterized by initial transmission capacity Q_{ij}^0, unit transmission cost e_t^{ij}, cost function of the transmission capacity increment, including fixed costs e_f^{ij} and variable costs $e_v^{ij}(Q_{ij}, Q_{ij}^0)$, e_v^{ij} is a monotonous convex function of increment $(Q_{ij} - Q_{ij}^0)$. The cost of the line expansion is the overnight construction cost amortized over the life-time T_{ij} of the line using discount rate r: $e^{ij} = r\frac{OC_{ij}}{1-e^{rT_{ij}}}$ (see [5] for the detailed discussion). Let q_{ij} denote the flow from the market i to market j, $q_{ij} = -q_{ji}$. Denote $Z(i)$ the set of nodes connected with node i. Under any fixed flows of the good $\overrightarrow{q} = (q_{ij}, (i,j) \in L)$ and production volumes $\overrightarrow{v} = (v_i,\ i \in N)$, the total social welfare for the network market is

$$W(\overrightarrow{q}, \overrightarrow{v}) = \sum_{i \in N} [U_i\left(v_i + \sum_{l \in Z(i)} q_{li}\right) - c_i(v_i)] - \sum_{(i,j) \in L,\quad i<j} E_{ij}(q_{ij}).$$

where

$$E_{ij}(q_{ij}) = \begin{cases} e_f^{ij} + e_v^{ij}\left(|q_{ij}| - Q_{ij}^0\right) + e_t^{ij}|q_{ij}|, & \text{if } |q_{ij}| > Q_{ij}^0, \\ e_t^{ij}|q_{ij}|, & \text{if } |q_{ij}| \le Q_{ij}^0. \end{cases}$$

The welfare optimization problem under consideration is

$$\max_{\overrightarrow{q}, \overrightarrow{v}} \ W(\overrightarrow{q}, \overrightarrow{v}) . \tag{1}$$

Let $\triangle S_i(p_i) = S_i(p_i) - D_i(p_i)$ denote the supply-demand balance.

Proposition 1 *Under any fixed flows $(q_{ij}, (i,j) \in L)$, for every $i \in N$, the optimal production volume at node i is $v_i = S_i(\widetilde{p_i})$, where $\widetilde{p_i}$ meets equation $\triangle S_i(\widetilde{p_i}) = \sum_{j \in Z(i)} q_{ij}$*

For any $\overline{L} \subseteq L$, consider a problem (2) with fixed set $\overline{L}$ of expanded lines. That is, $|q_{ij}| \leq Q^0_{ij}$ for $(i,j) \in L \setminus \overline{L}$, and the fixed costs are always included in E_{ij} for $(i,j) \in \overline{L}$.

Proposition 2 *The latter problem is convex, and its solution $(\overrightarrow{q}, \overrightarrow{v})(\overline{L})$ meets FOCs which determine the competitive equilibrium of the corresponding network market.*

Let $\widetilde{W}(\overline{L})$ denote the maximal welfare in the latter problem. Then problem (1) reduces to $\max_{\overline{L} \subseteq L} \widetilde{W}(\overline{L})$. Below we also consider problem (1) without construction costs and under constraint: $|q_{ij}| \leq Q_{ij}, (i,j) \in L$. Let $\widetilde{p_i}(\overrightarrow{Q}), i \in N$, denote the equilibrium prices corresponding to the solution of this problem.

Definition 1 *The model under consideration meets the flow structure invarience condition if, for any $\overrightarrow{Q} \geqslant \overrightarrow{Q}^0, (i,j) \in L$, $sign(p_i(\overrightarrow{Q}) - p_j(\overrightarrow{Q})) = sign(p_i(\overrightarrow{Q}^0) - p_j(\overrightarrow{Q}^0))$.*

A function $w(\overline{L}), \overline{L} \subseteq L$, is submodular (resp. supermodular) on L, if for any $L_1, L_2 \subseteq L$ $w(L_1) + w(L_2) \geqslant (\leqslant)\ w(L_1 + L_2) + w(L_1 \cap L_2)$. The desirable properties of the welfare function closely relate to the flow structure invariance condition. In general the function is neither submodular nor supermodular even for chain-type graphs, where $L = \{(i, i+1), i = 1, ..., n-1\}$. Consider a market with 3 nodes where $p_1(\overrightarrow{Q^0}) > p_2(\overrightarrow{Q^0}) > p_3(\overrightarrow{Q^0})$. Then the function is supermodular according to Theorem 1 given below. If flow directions converge, then the function is submodular by Theorem 2. In general a chain-type market may include both structures as its components and meet none of the conditions of super- or submodularity. Moreover, flow directions may change as the capacities increase. Below we establish conditions for the flow structure invariance and examine the welfare function for chain-type and star-type markets.

Theorem 2 *For a chain-type market with n nodes, let the initial prices $p_i(\overrightarrow{Q}^0)$, $i = 1, .., n$, monotonously decrease in i. Then, for any $\overrightarrow{Q} \geq \overrightarrow{Q}^0$, $p_i(\overrightarrow{Q}) \geq p_{i+1}(\overrightarrow{Q})$, $i = 1, .., n-1$, and function $\widetilde{W}(\overline{L})$ is supermodular. The complexity of search for the optimal set $\overline{L}^*$ under $\overrightarrow{Q}^0 = 0$ does not exceed $\frac{(n-1)n}{2}$.*

Consider a star-type market where $N = \{0, 1, .., n\}, L = \{(0, i), i = 1, .., n\}, p_i(\overrightarrow{Q}^0) < p_0(\overrightarrow{Q}^0)$ for $i \in I_1 = \{2, .., m\}, p_i(\overrightarrow{Q}^0) > p_0(\overrightarrow{Q}^0)$ for $i \in I_2 = \{m+1, .., n\}$. For $M \subseteq L$, let $(\overrightarrow{Q}^0 || \overrightarrow{Q}^\infty_M)$ denote vector $\overrightarrow{Q}$ such that $Q_l = Q^0_l$ for $l \notin M$, $Q_l = \infty$ for $l \in M$.

Theorem 3 *The market meets the condition of the flow structure invarience if and only if* $\forall i \in I_1 \ p_i(\overrightarrow{Q}^0 || \overrightarrow{Q}^\infty_{I_1}) < p_0(\overrightarrow{Q}^0 || \overrightarrow{Q}^\infty_{I_1})$ *and* $\forall i \in I_2 \ p_i(\overrightarrow{Q}^0 || \overrightarrow{Q}^\infty_{I_2}) > p_0(\overrightarrow{Q}^0 || \overrightarrow{Q}^\infty_{I_2})$. *Under this condition, the social welfare function* $\widetilde{W}(L_1 \cup L_2)$ *is submodular in* $L_1 \subseteq I_1$ *under a fixed set* $L_2 \subseteq I_2$, *and is also submodular in* $L_2 \subseteq I_2$ *under a fixed set* $L_1 \subseteq I_1$. *Besides that, for any* $L_1, l \in I_1 \setminus L_1$, *the welfare function increment* $\widetilde{W}(l \cup L_1, L_2) - \widetilde{W}(L_1, L_2)$ *monotonously increases in the set* L_2, *and for any* L_2, $l \in I_2 \setminus L_2$, *the increment* $\widetilde{W}(L_1, l \cup L_2) - \widetilde{W}(L_1, L_2)$ *monotonously increases in the set* L_1.

These properties of tree-type markets allow to use the known algorithms [3] for submodular and supermodular functions maximization in order to solve the optimization problem.

References

1. A. A. Vasin and E. A. Daylova, Optimum Throughput of a System of Product Logistics between Two Markets, Moscow University Computational Mathematics and Cybernetics, 2014, Vol. 38, No. 3, pp. 136-141., 2014.
2. Guisewite G.M., Pardalos P.M., Minimum concave-cost network flow problems: Applications, complexity, and algorithms. Annals of Operations Research, 25 (1); 1990. p. 75-99.
3. Khachaturov V.R. Mathematical Methods of Regional Programming. Nauka, Moscow; 1989 (in Russian).
4. Daylova E.A., Vasin A.A. Determination of Transmission Capacity for a Two-node Market. Procedia Computer Science. 31; 2014. p. 151–157.
5. Stoft S. Power System Economics: Designing Markets for Electricity. New York. Wiley; 2002.

Optimal regulation norms for competitive markets*

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This paper considers a competitive market of a homogeneous good with production negative externalities. We provide a theoretical model for determination of optimal regulation norms. Our study follows the approach that determines regulation norms proceeding from the social welfare maximization problem (see [1], [2], [3], [4]). We find out conditions for existence of a uniform optimal norm for all producers and provide an explicit formula for calculation of some sanitary norms.

Let $A = \{1, ..., n\}$ be a set of firms producing a homogeneous good. Its production concerns with some negative external effect. Besides production volume q^a the negative impact depends on technological parameter r^a established by the producer. Below we call it the *internal standard*. Thus, formally a strategy of producer a is a pair (q^a, r^a). Below we consider also an external norm related to the parameter. Production costs of producer a correspond to the following expression:

$$C^a(q^a, r^a) = \overline{c}^a(q^a) + c1^a(q^a, r^a) + c2^a(q^a, r^a), \tag{1}$$

where $\overline{c}^a(q^a)$ is the minimal cost of the volume q^a production, $c1^a(q^a, r^a)$ - the additional cost related to the normative standard, $c2^a(q^a, r^a)$ - the average cost of the negative effect's compensation under the given producer's strategy.

Consumers behavior is characterized by continuous demand function $D(p)$ with standard properties: $D(p)$ decreases and is differentiable almost everywhere, it is equal to zero when the price exceeds some level. The demand does not depend on standards set by producers because consumers do not have reliable information about them and, moreover, cannot estimate the impact of this factor on their utilities.

Let $\overline{r}$ denote an external norm established by some regulating government body. The norm is typically set for all producers of the good and does not depend on particular $a \in A$. We assume that it limits permitted internal standards from above and thus determines the set of possible strategies for each producer: pair (q^a, r^a) is feasible if $r^a \leqslant \overline{r}$. Consider a model where the market is perfectly competitive and each producer aims to maximize his profit under given norm and price:

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$$(q^{a*}, r^{a*})(p, \overline{r}) \to \max_{(q^a, r^a): r^a \leqslant \overline{r}} (pq^a - C^a(q^a, r^a)). \qquad (2)$$

The supply function of producer a is determined as

$$S^a(p, \overline{r}) = Arg \max_{q^a} (pq^a - \min_{r^a \leqslant \overline{r}} C^a(q^a, r^a)). \qquad (3)$$

The total supply function is $S(p, \overline{r}) = \sum_a S^a(p, \overline{r})$, and the competitive equilibrium price $\widetilde{p}(\overline{r})$ proceeds from the condition $D(p) \in S(p, \overline{r})$.

Proposition 1 *Assume that, for each producer a, his cost function may be represented as $C^a(q^a, r^a) = \overline{c}^a(q^a) + q^a \overline{\overline{c}}^a(r^a)$, where $\overline{c}^a(q)$ is a convex and increasing function, $\overline{\overline{c}}^a(r)$ is a convex function that reaches its minimal value at $\hat{r}^a$. Then the equilibrium price $\widetilde{p}(\overline{r})$ does not increase in $\overline{r}$.*

Thus, the tougher the norm the grater is the price. Below we discuss the following issues: what is the optimal state of the economy with account of the negative externality? How to reach this optimal state by means of the regulation?

Consider the optimal strategy of producer a at the equilibrium under a given norm: $\overline{r}$: $q^{a*}(\overline{r}) \in S^a(\widetilde{p}(\overline{r}), \overline{r})$, $r^{a*}(\overline{r}) = \arg\min_{r^a \leqslant \overline{r}} C^a(q^{a*}(\overline{r}), r^a)$. The social welfare with account of the negative externality is determined as

$$W(\overline{r}) =$$

$$= \int_0^{D(\widetilde{p}(\overline{r}))} D^{-1}(q)dq - \sum_a C^a(q^{a*}(\overline{r}), r^{a*}(\overline{r})) - \sum_a C^a_{lost}(q^{a*}(\overline{r}), r^{a*}(\overline{r})),$$

$\int_0^{D(\widetilde{p}(\overline{r}))} D^{-1}(q)dq$ is the total consumers' utility without the impact of the negative externality, $\sum_a C^a(q^{a*}(\overline{r}), r^{a*}(\overline{r}))$ shows the total costs of producers, and C^a_{lost} is the loss of the social welfare related to the negative externality that is not compensated by the producer.

Consider a problem of the social welfare optimization for this economy under a centralized planning. Let $C^a_W = C^a + C^a_{lost}$. Then the problem may be set as follows:

$$\int_0^{\sum_a q^a} D^{-1}(q)dq - \sum_a C^a_W(q^a, r^a) \to \max_{q^a, r^a, a \in A}. \qquad (4)$$

Proposition 2 *Assume that function $\overline{C}^a_W(q^a) \stackrel{def}{=} \min_{r^a} C^a_W(q^a, r^a)$ is convex. Then a combination of solutions (q^{a*}, r^{a*}) for problem (2) under constraint $r^a \leqslant \hat{r}^a$, $a \in A$, is a solution of problem (4).*

If the optimal values r^a in the solution of the problem (4) are all equal to $\hat{r}$, then we call $\hat{r}$ a *uniform optimal norm.*

Consider a particular case where the norm bounds concentration of some harmful substance in the purchased good. For each producer A, let $\overline{c}^a(q^a)$ denote the cost of production of the given volume without any purification. The initial concentration of the substance is r^a_0, and function $c^a_{marg}(r)$ determines the marginal cost of purification depending on the concentration. Then the cost of purification under standard normative r^a is $c1^a(q^a, r^a) = q^a \int_{r^a}^{r^a_0} c^a_{marg}(r)dr$, where function c^a_{marg} decreases in r. This property holds because reduction of the concentration in a given amount is the cheaper the higher is the initial concentration.

Under a soft internal standard a producer faces the risk of additional costs related to compensations of losses for consumers which suffered from high concentration of the substance. Let $w(r^a)$ denote the money equivalent of the average consumer loss per one unit of the good. This function monotonously increases in r^a, as well as the share $\pi^a(r^a)$ of the loss that the producer compensates to consumers. Thus, the total production costs meet $C^a(q^a, r^a) = \overline{c}^a(q^a) + c1^a(q^a, r^a) + c2^a(q^a, r^a)$ where $c2^a(q^a, r^a) = q^a \pi^a(r^a) w(r^a)$.

According to equation (2), the internal standard of producer a meets condition

$$r^{a^*} = \arg\min_{r^a} \left(\int_{r^a}^{r^a_0} c^a_{marg}(r)dr + \pi^a(r^a)w(r^a) \right).$$

In the perfectly competitive market the optimal strategy under a given norm $\overline{r}$ is a solution of the problem

$$(q^{a^*}, r^{a^*})(p, \overline{r}) \to$$
$$\to \max_{\substack{(q^a, r^a): \\ r^a \leqslant \overline{r}}} \left(pq^a - \overline{c}^a(q^a) - q^a \int_{r^a}^{r^a_0} c^a_{marg}(r)dr - q^a \pi(r^a) w(r^a) \right).$$

The total of production costs and consumers' losses in this case is

$$C^a_W(q^a, r^a) = \overline{c}^a(q^a) + q^a \int_{r^a}^{r^a_0} c^a_{marg}(r)dr + q^a w(r^a), \qquad (5)$$

and the uncovered consumers' losses related to the harmful substance are $C^a_{lost}(q^a, r^a) = q^a(1 - \pi(r^a))w(r^a)$. Denote $\sum_a q^a = q_\Sigma$. Then the social welfare maximization problem is

$$\int\limits_0^{q_\Sigma} D^{-1}(q)dq - \sum_a \left[\overline{c}^a(q^a) + q^a \int\limits_{r^a}^{r_0^a} c^a_{marg}(r)dr + q^a w(r^a)\right] \to \max_{q^a,\, r^a,\, a \in A}.$$

Proposition 3 *Assume that the purification technology characterized by the marginal cost function $c_{marg}(r)$ is the same for all producers. Then the optimal sanitary norm r^* binding the maximal concentration of the harmful substance meets equation $c_{marg}(r^*) = w'(r^*)$, and the optimal production volumes proceed from the system*

$D^{-1}(q_\Sigma) = \overline{c}^{a'}(q^a) + \int_{r*}^{r_0^a} c^a_{marg}(r)dr + w(r^*), a \in A.$

References

1. Atkinson A.B. and Stiglitz J.E. Lectures on Public Economics // McGraw-Hill, 1980.
2. Laffont J.J. and Tirole J. A Theory of Incentives in Procurement and Regulation // MIT Press, 1993.
3. Spulber D.F. Effluent regulation and long-run optimality // Journal of Environmental Economics and Management, Vol. 12, (1985) pp. 103-116.
4. Polterovich V. Elements of the Reform Theory // Moscow, Ekonomika, (2007)

Predictive models for congested traffic

4-step forecasting transport model with trip chaining behaviour*

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We consider the problem of modeling and forecast of traffic and passenger flows in a large city. The standard approach to solving this problem is a 4-step scheme [1, 2], which includes (1) trip generation, (2) trip distribution, (3) modal split, and (4) traffic assignment. The main advantage of this approach is the simplicity of data preparation and software implementation and relatively low consumption of computer resources, which allows for large scale network modeling. However, the standard 4-step scheme does not account for some important aspects of travel behaviour, one of which is the interrelationship between trips, that form chains of trips. We presents a combined approach, which allows to take into account a major impact of trip chains while maintaining the computational simplicity of the 4-step scheme.

The movements of people form a chains that start and end at the same place, usually at home. Various mobility surveys show that the most common trip chains are chains with single destination *Home* $\rightarrow$ *Object* $\rightarrow$ *Home* and chains of three trips *Home* $\rightarrow$ *Object 1* $\rightarrow$ *Object 2* $\rightarrow$ *Home*. One more fairly common chain *Home* $\rightarrow$ *Object 1* $\rightarrow$ *Object 2*

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$\rightarrow$ *Object 1* $\rightarrow$ *Home* is splitting on two simple chains: *Home* $\rightarrow$ *Object 1* $\rightarrow$ *Home* and *Object 1* $\rightarrow$ *Object 2* $\rightarrow$ *Object 1*. Other trip chains in the demand structure can be neglected.

The set of trip chains with certain purposes in certain periods of day (early morning, morning peak, midday off-peak, evening peak, late evening, night) will be referred to as demand element (for instance, $Home \xrightarrow{Morning\ peak} Work \xrightarrow{Evening\ peak} Home$). We evaluate distribution of trip chains over demand elements based on various mobility researches.

The calculation of trip matrices includes calculation of daily matrices for each demand stratum, followed by calculation of hourly mode-specific matrices (by foot, by car and by public transport) for each time period [3]. We assume that people usually do not change mode during the chain of trips. Thus will apply the same splitting coefficients to all trips in a single chain. These coefficients are evaluated separately for demand elements.

Modal split coefficients depend on the *generalized travel costs* for different modes. Travel costs are composed of the following components:

- for private transport:
  - starting time (assigned to connectors from zones),
  - travel time (road links and turns),
  - operating costs (road links),
  - toll roads fee (road links),
  - parking fee in certain areas (connectors to zone).
- for public transport:
  - waiting and boarding time (boarding links),
  - travel time (according to a time tables),
  - the fare payment (a fixed payment or a distance-dependent payment).

For evaluation of the modal split we divide the population into two classes based on car ownership (access to a car). Thus car owners have a choice of three modes, while the others have only two alternatives (excluding car). The proportion of populations of these classes varies over the territory of modelling.

We also use a similar modeling framework for freight transport, which include the following steps (for each class of tracks):

1. Estimation of the total daily trips produced and attracted by a zone for each demand stratum.

2. Calculation of daily matrices for each demand stratum.

3. Calculation of hourly matrices, taking into account the restrictions of entry and moving of trucks of certain classes in certain areas of the city, applied at certain time periods.

To implement this freight transport modeling framework the following inputs are required:

- freight demand structure description:
  - freight trips generators and attractors classification (parking stations, warehouses, factories, malls, shops, etc.),
  - freight trip chains description (including intermediate trips multiplicity),
  - trip chains distribution by truck type (light, medium and heavy trucks),
  - trip distribution by time of day for each trip chain,
- generators/attractors spatial distribution with their attributes.

The proposed modeling framework was implemented for the traffic model of the Moscow agglomeration. Model calibration was based on a hierarchical data structure [4], which implies step-by-step calibration, starting with daily citywide indicators and then moving towards details on the time of day, city zones, etc. A databank for calibration of the Moscow traffic model includes:

- traffic counts on roads;
- passenger counts at subway and suburban railway stations;
- passenger counts at bus stops near subway stations;
- average travel times of typical routes at different time periods of a day.

References

1. Shvetsov V.I. Mathematical modeling of traffic flows // Automation and Remote Control. 2003. No 11. P. 3–46.

2. Ortuzar J. de D. and Willumsen L.G. Modelling Transport. Wiley, 2011.
3. Aliev A.S., Mazurin D.S., Maksimova D.A., and Shvetsov V.I. The structure of the complex model of the transport system of Moscow // Proceedings of the ISA RAS. 2015. V. 65, No 1. P. 3–15. (in russian)
4. Mazurin D.S. and Shvetsov V.I. Data structure for city traffic model calibration // Proceedings of the ISA RAS. 2015. V. 65, No 1. P. 16–23. (in russian)

The traffic flow simulation in a growing urban infrastructure with using a tool set for creating interactive virtual environments

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Mathematical equilibrium models and the specialized software based on them are one of the effective tools to support managerial decisions in the transportation planing. Such models consider the traffic flow as a entire unit and make it possible to predict the traffic volume and traffic assignment in the network with flow-dependent travel costs.

Predictive modelling of traffic flows consists of solving the following problems [1]: 1) trip generation; 2)trip distribution; 3) model split; 4) route assignment. The problems are considered in succession, the output from one problem is being the input to the next one. In order to achieve an agreement between the results of problem solutions the process have to be repeated many times.

The forecast congestion of the transportation network is determined at the fourth step. The basic assumption concerning the way the network users choose their routes is usually made according to the so-called Wardrop's first behavioural principle: drivers use only routes corresponding to minimal travel costs [2].

Despite the many advantages the current software of traffic prediction has two drawbacks 1) it does not contain implementations of recent advances of the mathematical modelling of traffic flows; 2) it requires preliminary training to be installed, supported and used.

In this paper the concept of the cloud service for interactive modelling of transport flows in a growing city infrastructure will be described. The main purpose of the service is operative evaluation of the network congestion as a result of various modifications of network elements and changes in the arrangement and designation of town planning objects.

The forecast of traffic flows is realized on the bases of the mathematical model which is the result of synthesis of the gravity model of description of trip distributions [3] and multimodal network equilibrium problem with elastic demand [4]. Equilibrium traffic flow pattern is defined as the solution of the following variational inequality

$$F(x^*)(x-x^*) - \frac{1}{\lambda} \sum_{\substack{m\,\in\,\mathcal{M}\\(i,\,j)\,\in\,\mathcal{O}\,\times\,\mathcal{D}}} \ln\left(\frac{o_i d_j \rho^*_{mij}}{(\sum\limits_{m\in\mathcal{M}} \rho^*_{mij})^2}\right)(\rho_{mij}-\rho^*_{mij}) \geq 0,$$

$$(x,\rho)\in\Omega=\left\{(x,\rho)\geq 0: \sum_{p\in P_{ij}} x_{mp}=\rho_{mij}, \quad m\in\mathcal{M},\ (i,j)\in\mathcal{O}\times\mathcal{D},\right.$$

$$\left.\sum_{j\in\mathcal{D}}\sum_{m\in\mathcal{M}}\rho_{mij}=o_i, \quad \sum_{i\in\mathcal{O}}\sum_{m\in\mathcal{M}}\rho_{mij}=d_j, \quad (i,j)\in\mathcal{O}\times\mathcal{D}\right\},$$

where $\mathcal{M}$, $\mathcal{O}$ and $\mathcal{D}$ are the sets of modes, origins and destinations, P_{ij} is the set of alternative routes for OD-pair $(i,j)\in\mathcal{O}\times\mathcal{D}$, $x=(x_{mp}: m\in\mathcal{M},\ p\in P_{ij},\ (i,j)\in\mathcal{O}\times\mathcal{D})$ and $F(x)=(F_{mp}(x): m\in\mathcal{M},\ p\in P_{ij},\ (i,j)\in\mathcal{O}\times\mathcal{D})$ are the route flow vector and the travel cost mapping, $\rho=(\rho_{mij}: m\in\mathcal{M},\ (i,j)\in\mathcal{O}\times\mathcal{D})$ is the correspondence matrix, o_i and d_j are the total number of trips generated by the origin $i\in\mathcal{O}$ and absorbed by the destination $j\in\mathcal{D}$, $\lambda>0$ is the calibration coefficient.

The solution of the variational inequality substitutes the last three stages of the four-phases iterative process of traffic modelling, which, in turn, improves calibration of the calculations and leads to more reliable results of traffic modelling. The assumption that the travel cost $F_{mp}(x)$ is the function of the load across the entire network allows us to capture supplementary flow relationships such as interactions among vehicles on different road links and turning priorities in junctions and etc.

A tool set for 3D visualization and traffic flows modeling is implemented. The tool set is a cloud service, which consists of three modules: a simulation module, a control module, and a visualization module.

The simulation module is realized on a high-performance server platform, control and visualization modules are realized on the IACPaaS cloud platform [5]. Communication between the platforms based on asynchronous dynamic http-queries.

The simulation module results are transmitted to the control module, whose main tasks are: processing, analysis and transmission of information between the modules in specific for each module formats. Analysis of the data in the control module is carried out using a virtual environment model [6]. The virtual environment model has a declarative representation. Processing and analysis results are transmitted to the visualization module in the same declarative format. The main component of the visualization module is interpreter. It provides 3D visualization and a program logic using the virtual environment model.

References

1. Patriksson M. The traffic assignment problem: models and methods. Utrecht, The Netherlands: VSP. 1994.
2. Wardrop J. Some theoretical aspects of road traffic research // Proceedings of the institute of Engineers. Part II. 1952. V. 1. P. 325–378.
3. Erlander S., Stewart N.T. The gravity model in transportation analisis: theory and extensions. Utrecht, The Netherlands: VSP. 1990.
4. Dafermos S. The general multimodal network equilibrium problem with elastic demand // Networks. 1982. V. 12, No 1. P. 57–72.
5. Gribova V.V, Kleschev A.S., Krylov D.A., Moskalenko F.M., Smagin S.V., Timchenko V.A., Tyutyunnik M.B., Shalfeeva E.A. Research project IACPaaS. Extensible information and software complex for development, control, and usage of intelligent software based on cloud computing // Iskusstvennyi intellekt i prinyatie reshenii. 2011. No. 1. pp. 27–35 (in Russian).
6. Gribova V.V., Fedorischev L.A. Virtual teachware and tools for its creation // Vestnik informacionnyh i komp'yuternyh tehnologii. 2012. No. 3. pp. 48–51 (in Russian).

Intermediate universal gradient method with inexact oracle*

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We consider the following convex composite optimization problem [1]:

$$F(x) = f(x) + h(x) \to \min_{x \in Q}.$$

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Definition 1. [1] Let function f be convex on convex set Q. We say that it is equipped with *a first-order* (δ, L)*-oracle* if for any $y \in Q$ we can compute a pair $(f_{\delta,L}(y), g_{\delta,L}(y))$, such that for all $x \in Q$

$$0 \le f(x) - (f_{\delta,L}(y) + \langle g_{\delta,L}(y), x - y\rangle) \le \frac{L}{2}\|x-y\|^2 + \delta.$$

Constant δ will be called *accuracy* of the oracle. A function $h(x)$ have simple structure and it's easy to compute it without an oracle.

Statement 1. [1] *Composite fast gradient method(FGM) Yu.E.Nesterov with* (δ, L)*-oracle converges with*

$$F\left(y^N\right) - F_* \le \varepsilon, \quad N = \mathrm{O}\left(\sqrt{\frac{LR^2}{\varepsilon}}\right), \quad \delta \le \mathrm{O}\left(\frac{\varepsilon}{N}\right).$$

where (N – a number of calling oracle). Up to constant estimations are optimal

Statement 2. [1] *We introduce oneparametric class with parameter* $p \in [0,1]$) *of intermediate gradient methods with such convergence rate*

$$F\left(y^N\right) - F_* \le \varepsilon, \quad N = \mathrm{O}\left(\left(\frac{LR^2}{\varepsilon}\right)^{\frac{1}{p+1}}\right), \quad \delta \le \mathrm{O}\left(\frac{\varepsilon}{N^p}\right). \tag{1}$$

Statement 3. [1] *Let*

$$\|\nabla f(y) - \nabla f(x)\|_* \le L_\nu \|y - x\|^\nu \tag{2}$$

with some $\nu \in [0,1]$. *Then*

$$0 \le f(y) - f(x) - \langle \nabla f(x), y - x\rangle \le \frac{L}{2}\|y-x\|^2 + \delta,$$

где $L = L_\nu \cdot \left[\frac{L_\nu}{2\delta}\frac{1-\nu}{1+\nu}\right]^{\frac{1-\nu}{1+\nu}}$.

Statement 4. *From* (1) *For the intermediate gradient method we derive such convergence rate* [2]

$$F\left(y^N\right) - F_* \le \varepsilon, \quad N = \mathrm{O}\left(\inf_{\nu\in[0,1]}\left(\frac{L_\nu R^{1+\nu}}{\varepsilon}\right)^{\frac{2}{1+2p\nu+\nu}}\right),$$

where $\delta \le \mathrm{O}\left(\frac{\varepsilon}{N^p}\right), p \in [0,1]$.

Proof For inexact oracle:

$$0 \leq f(y) - f(x) - \langle \nabla f(y) - \nabla f(x) \rangle \leq \frac{L}{2} \|y - x\|^2 + \delta$$

we have estimation

$$N = O\left(\left(\frac{LR^2}{\varepsilon} \right)^{\frac{1}{p+1}} \right), \delta \leq O\left(\frac{\varepsilon}{N^p} \right).$$

Let use the notion of (δ, L)-oracle for solving the problems with exact first-order information but with a lower level of smoothness.

$$L = L_\nu \left[\frac{L_\nu}{2\delta} \frac{1-\nu}{1+\nu} \right]^{\frac{1-\nu}{1+\nu}}.$$

It means that our estimation changes such way:

$$N = O\left(\left(\frac{LR^2}{\varepsilon} \right)^{\frac{1}{p+1}} \right) = O\left(\left(\frac{R^2}{\varepsilon} L_\nu \left[\frac{L_\nu}{2\delta} \frac{1-\nu}{1+\nu} \right]^{\frac{1-\nu}{1+\nu}} \right)^{\frac{1}{p+1}} \right) =$$

$$= O\left(\left(R^2 \varepsilon^{-1} L_\nu^{\frac{2}{1+\nu}} \delta^{\frac{\nu-1}{1+\nu}} \right)^{\frac{1}{p+1}} \right) \Rightarrow$$

$$N^{p+1} \sim R^2 \varepsilon^{-1} L_\nu^{\frac{2}{1+\nu}} \delta^{\frac{\nu-1}{1+\nu}} \sim R^2 \varepsilon^{-1} L_\nu^{\frac{2}{1+\nu}} \left(\frac{\varepsilon}{N^p} \right)^{\frac{\nu-1}{1+\nu}} \sim$$

$$\sim R^2 \varepsilon^{\frac{-2}{1+\nu}} L_\nu^{\frac{2}{1+\nu}} N^{-\frac{p\nu-p}{1+\nu}} \Rightarrow$$

$$N^{p+1+\frac{p\nu-p}{1+\nu}} \sim L_\nu^{\frac{2}{1+\nu}} R^2 \varepsilon^{\frac{-2}{1+\nu}} \Rightarrow$$

$$N^{\frac{p\nu+p+\nu+1+p\nu-p}{1+\nu}} \sim \left(\frac{L_\nu R^{1+\nu}}{\varepsilon} \right)^{\frac{2}{1+\nu}} \Rightarrow$$

$$N^{1+2p\nu+\nu} \sim \left(\frac{L_\nu R^{1+\nu}}{\varepsilon} \right)^2 \Rightarrow$$

$$N \sim \left(\frac{L_\nu R^{1+\nu}}{\varepsilon} \right)^{\frac{2}{1+2p\nu+\nu}} \Rightarrow$$

$$F(y^N) - F_* \leq \varepsilon, N = O\left(\left(\frac{L_\nu R^{1+\nu}}{\varepsilon}\right)^{\frac{2}{1+2p\nu+\nu}}\right), \delta \leq O\left(\frac{\varepsilon}{N^p}\right). \blacksquare$$

This method has a very good application in transport problems. We could use it to solving dual optimization problem in searching equilibria in mixed models of flow distribution in large transport networks.

References

1. Devolder O., Glineur F., Nesterov Yu.E. "First-order methods of smooth convex optimization with inexact oracle."Springer, 2013
2. Gasnikov A., and Kamzolov D., Mendel M. "Universal composite prox-method for strictly convex optimization problems"2016. http://arxiv.org/abs/1603.07701
3. Anikin A.S., Gasnikov A.V., Semenov V.V. "Parallelizability dual method for searching equilibria in mixed models of flow distribution in large transport networks."ORM 2016

Empirical synchronized flow in oversaturated city traffic

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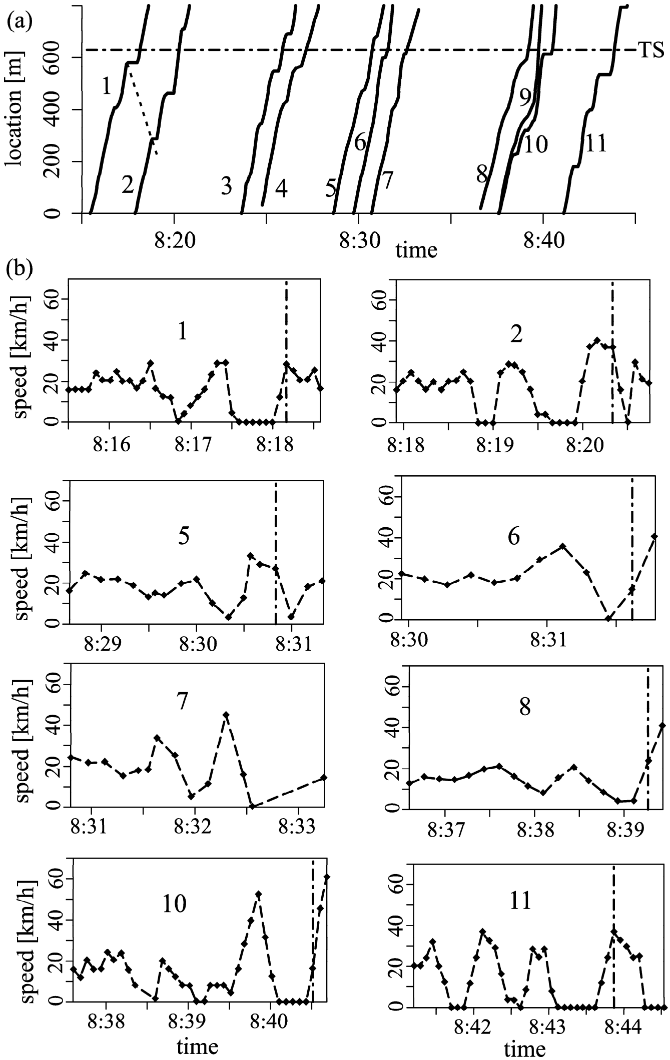
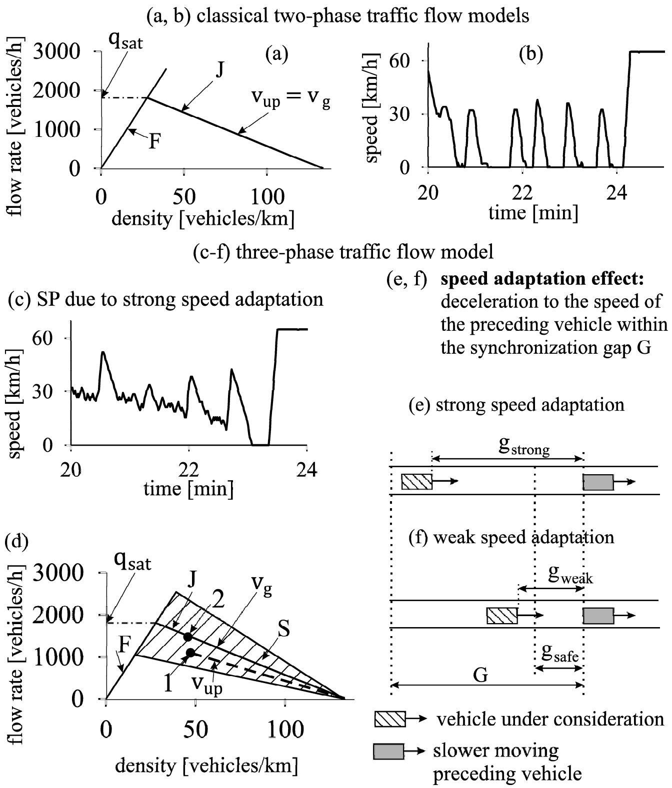
Based on a study of anonymized GPS probe vehicle traces measured by personal navigation devices (PND) in vehicles randomly distributed in city traffic, empirical synchronized flow in oversaturated city traffic has been revealed. It turns out that real oversaturated city traffic resulting from speed breakdown in a city in most cases can be considered random spatiotemporal alternations between regular sequences of moving queues and synchronized flow patterns (SP) in which the moving queues do not occur. This work relies on the results in [1].

Conclusions: In real oversaturated city traffic caused by speed breakdown, the following empirical microscopic spatiotemporal traffic



Fragment of typical empirical microscopic spatiotemporal structure of oversaturated city traffic: (a) Vehicle trajectories of probe vehicles on road section measured on February 05, 2013. (b) Microscopic (single-vehicle) speeds (black squares) along vehicle trajectories shown by the same numbers in (a). Dashed-dotted lines show traffic signal location in (a) and time instances of vehicle passing the signal in (b).



Explanations of oversaturated traffic in classical theory (a, b) [3] and three-phase theory (c–f) [4]: (b, c) Simulations of speed in moving queues (b) and SPs (c). J – line J, q_{sat} is a saturation flow rate, F – free flow, S – synchronized flow.

patterns have been revealed: (i) Empirical synchronized flow patterns (SP). (ii) Classical regular sequences of moving queues. (iii) Random spatiotemporal alternations between regular sequences of moving queues and SPs. (iv) Simultaneous occurrence of SPs and moving queues in different road lanes. Empirical probability of speed breakdown in city traffic is well-described by a theoretical one found in [2].

References

1. B. S. Kerner, P. Hemmerle, M. Koller, G. Hermanns, S. L. Klenov, H. Rehborn, M. Schreckenberg, Phys. Rev. E **90** 032810 (2014).
2. B.S. Kerner, Phys. Rev. E **84**, 045102(R) (2011); B.S. Kerner, Europhys. Lett. **102**, 28010 (2013); B.S. Kerner, Physica A **397** 76–110 (2014).
3. F.V. Webster, Road Research Technical Paper No. 39, Road Research Laboratory, London, UK (1958); G.F. Newell, SIAM Review, **575**, 223–240 (1965).
4. B.S. Kerner, S.L. Klenov, G. Hermanns, P. Hemmerle, H. Rehborn, and M. Schreckenberg, Phys. Rev. E **88**, 054801 (2013).

Phase transitions in deterministic traffic flow models

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Theoretical modelling and computer simulation of transportation systems is a very popular field, see very impressive review [2]. There are two main directions in this research - macro and micro models. Macro approach does not distinguish individual transportation units and uses analogy with the fluid flow in hydrodynamics, see [1]. Stochastic micro models are most popular and use almost all types of stochastic processes: mean field, queueing type and local interaction models. We consider here completely deterministic transportation flows. Although not as popular as stochastic traffic, there is also a big activity in this field, see [3,4,5,6]. In these papers interesting results are obtained for sufficiently general protocols.

Here we follow another strategy: for simplest possible protocols we try to get results as concrete as possible. Namely, we consider the one-way road traffic model organized as follows.

At any time $t \geq 0$ there is finite or infinite number of point particles (may be called also cars, units etc.) with coordinates $z_k(t)$ on the real axis, enumerated as follows

$$... < z_n(t) < ... < z_1(t) < z_0(t) \tag{1}$$

We assume that the rightmost car (the leader) moves "as it wants", that is the trajectory $z_0(t)$ is often assumed to have nonnegative velocity.

Our problem is to find the simplest possible local protocol (control algorithm) which would guarantee both safety (no collisions), stable (or even maximal) density of the flow or maximal current. Otherwise speaking, we try to find control mechanism which guarantees that the distance between any pair of neighbouring cars is close (on all time interval $(0, \infty)$) to some (given a priori) fixed number, that defines the density of the flow.

More exactly, denoting $r_k(t) = z_{k-1}(t) - z_k(t)$, and

$$I = \inf_{k \geqslant 1} \inf_{t \geqslant 0} r_k(t), \quad S = \sup_{k \geqslant 1} \sup_{t \geqslant 0} r_k(t),$$

we try to get the bounds - lower positive bound on I and upper bound on S - as close as possible.

Locality (of the control) means that the "driver" of the k-th car, at any time t, knows only its own velocity $v_k(t)$ and the distance $r_k(t)$ from the previous car. Thus, for any $k \geq 1$ the trajectory $z_k(t)$, being deterministic, is uniquely defined by the trajectory $z_{k-1}(t)$ of the previous particle.

Using physical terminology one could say that if, for example, $r_k(t)$ becomes larger than d, then some virtual force F_k increases acceleration of the particle k, and vice-versa. Thus the control mechanism is of the physical nature, like forces between molecules in crystals but our "forces" are not symmetric. Thus our system is not a hamiltonian system. Nevertheless, our results resemble the dynamical phase transition in the model of the molecular chain rapture under the action of external force, see [7]. However here we do not need the double scaling limit used in [7].

We will see however that for the stability, besides F_k, also friction force $-\alpha v_k(t)$, restraining the growth of the velocity $v_k(t)$, is necessary, where the constant $\alpha > 0$ should be chosen appropriately. Taking F_k to be simplest possible

$$F_k(t) = \omega^2 (z_{k-1}(t) - z_k(t) - d) \tag{2}$$

we get that the trajectories are uniquely defined by the system of equations for $k \geq 1$

$$\frac{d^2 z_k}{dt^2} = F_k(t) - \alpha \frac{dz_k}{dt} = \omega^2 (z_{k-1}(t) - z_k(t) - d) - \alpha \frac{dz_k}{dt} \tag{3}$$

Stability depends not only on the parameters α, ω, d but also on the initial conditions and on the movement of the leader (on its velocity and acceleration). This is easy to understand for the case of $N + 1$ particles. For example, for $N = 1$, where the calculations are completely trivial, assume also the simplest leader movement

$$z_0(t) = vt, t \geq 0 \tag{4}$$

Then, if initial condition for the second particle are

$$z_1(0) = -a = -(d + \frac{\alpha}{\omega^2} v), \dot{z}_1(0) = v,$$

then $z_1(t) = -a + vt$ for any d, α, ω. However, if we change only the initial velocity $\dot{z}_1(0) = w$ to some $w > 0$, then for any α, ω there exists $w_1 = w_1(\alpha, \omega, d)$ such that for any $w \geq w_1$ collision occurs.

For $N = 2, 3, ...$ the situation becomes more and more complicated, and its study has no much sense. That is why we study, in the space of two parameters α, ω (for fixed d), stability conditions, which are uniform in N and in large class of reasonable initial conditions and reasonable movement of the leader.

Natural (reasonable) initial conditions are as follows: at time 0 it should be

$$0 < \inf_{k \geqslant 1} r_k(0) \leq \sup_{k \geqslant 1} r_k(0) < \infty$$

As for the leader movement, it is sometimes sufficient to assume that the function $z_0(t)$ were continuous, but in other cases it is assumed to twice differentiable and has the following bounds on the velocity and acceleration of the leader:

$$\sup_{t \geqslant 0} |\dot{z}_0(t)| = v_{\max}, \quad \sup_{t \geqslant 0} |\ddot{z}_0(t)| = a_{\max}, \tag{5}$$

It appears that under these conditions there are 3 sectors in the quarter-plane $R_+^2 = \{(\alpha, \omega)\}$: 1) $\alpha > 2\omega$, where we can prove stability, 2) $\alpha < \sqrt{2}\omega$, where we can prove instability, and the sector 3) $\sqrt{2}\omega \leq \alpha \leq 2\omega$, where we can prove stability only for more restricted classes of initial conditions and of the leader motion.

References

1. Prigogine I., Herman R. Kinetic theory of vehicular traffic. //New York: Elsevier, 1971.
2. Helbing D. Traffic and related self-driven many particle systems. // Rev. Mod. Phys. 73, 2001. P. 1067–1141.
3. Feintuch A., Francis B. Infinite chains of kinematic points. //Automatica 48, 2012. P. 901–908.
4. Qing Hui, Jordan M. Berg. Semistability theory for spatially distributed systems. //Proceedings of the IEEE Conference on decision and control, January 2009.
5. Melzer S.M., Kuo B.C. Optimal regulation of systems described by a countably infinite number of objects. //Automatica, 1971. V. 7, P. 359–366. Pergamon Press.
6. Swaroop D., Hedrick J.K. String stability of interconnected systems. //IEEE transactions on automatic control, March 1996. V. 41, № 3.
7. Malyshev V.A., Musychka S.A. Dynamical phase transition in the simplest molecular chain model. //Theoretical and mathematical physics, 2014. V. 179, № 1, P. 123–133.

Computer simulation of traffic flow and mathematical description

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There are a lot of different approaches in the traffic flow theory. One of the most popular is macroscopic approach. Under certain conditions a traffic flow can be considered as a flux of special particles. We denote density and velocity of traffic flow on a neighbourhood of x at the moment t as $\rho(x,t)$ and $v(x,t)$, respectively. The value of flow $q(x,t)$ is the average amount of vehicles, that passed throw the point x in the unitary interval of time (for example an hour) at the moment t. These quantities are related by the conservation law, the continuity of flow and the equation of state. They are similar with ones from the hydrodynamics. The car velocity should be high if the density is low, and controversially otherwise. Therefore we can assume that velocity is a certain steadily decreasing function of the density

$$v = V(\rho), \qquad V = V(\rho)\downarrow, \qquad 0 \leqslant \rho \leqslant \rho_{\max}. \tag{1}$$

Here $\rho_{\max}$ is the value corresponds to a traffic jam. One of the most important relation is

$$q = Q(\rho), \qquad 0 \leqslant \rho \leqslant \rho_{\max}, \tag{2}$$

which called *the fundamental diagram*. This dependence between the flow value and the density plays the key role in the traffic flow theory.

In this report we discuss some problems connected with reconstruction of the dependence (1) and (2) using computer simulation. We introduce a program "Cars" that simulates a traffic flow using microscopic approach. Therefore, every vehicle is treated as an individual object. It has the set of parameters, such as length, maximum speed, maximum and minimum acceleration and deceleration and so on. All these parameters correspond with the real values. Every car moves using the same algorithm that prevents a colliding but allows to move as quickly as possible. The algorithm handles the data available to an "ordinary" driver. The control parameter is the acceleration of the vehicle. The configuration of the road is a single–lane one–way road. We also examine a ring road that allows us to study an autonomous clusters of cars. Using our program we can obtain a numerous amount of data. Then we use these data for establishing connections between variables and for revealing some typical phenomena.

We emphasize some thematics of our research:

1. Connection between macroscopic quantities ρ, v, q with microscopic parameters of individual cars;

2. Ocular demonstration of mathematical effects which exist in quasi-linear equation theory (strong discontinuity, shock waves, Rankine–Hugoniot condition, bifurcation of solutions and etc.)

3. Forming of the traffic jams which move backwards.

Many of discussed phenomena were mentioned in previous papers [1–6].

References

1. Gasnikov A. V. i dr. Vvedenie v matematicheskoe modelirovanie transportnykh potokov: Uchebnoe posobie [Introduction in mathematical simulation of traffic flows]/ Pod red. A. V. Gasnikova. Izdanie 2-e, ispr. i dop.—M.: MCNMO, 2013.—427 p.
2. Lighthill M. J., Whitham G. B. On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads // Proceedings of the Royal

Society of London. Series A, Mathematical and Physical Sciences.—1955.—Vol. 229, No 1178.—P. 317–345.

3. Nagel K., Schreckenberg M. A cellular automaton model for freeway traffic // Journal de Physique I France.—1992.—Vol. 2, No 12.—P. 2221–2229.
4. Smirnov N. N., Kiselev A. B., Nikitin V. F., Umashev M. V. Matematicheskoe modelirovanie avtotransportnykh potokov [Mathematical simulation of traffic flows].—M.: MGU, 1999.—31 p.
5. Lax P. D. Hyperbolic Partial Differential Equations (Courant Lecture Notes). Courant Institute of Mathematical Sciences. — NY: American Mathematical Society, 2006 — 217 p.
6. Evans L.C. Partial Differential Equations. — (Graduate Studies in Mathematics. Vol. 19). American Mathematical Society, 2010. — 749 p.

Asymptotic analysis of complex stochastic systems

Limit theorems for multichannel queuing systems with abandonments

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We consider queuing systems with r heterogeneous channels.The service time η_n^i of the n-th customer by the i-th server has distribution function $B_i(x)$ with finite mean β_i^{-1}. Let $\beta = \sum_{i=1}^r \beta_i$. Customers are served in order of their arrivals at the system. Service times of customers are independent random variables.

The input flow $X(t)$ is assumed to be regenerative. Let θ_i be the i-th regeneration point of $X(t)$, $\tau_i = \theta_i - \theta_{i-1}$, $\xi_i = X(\theta_i) - X(\theta_{i-1})$ $(i = 1, 2, \dots; \theta_0 = 0)$. Then τ_i is the regeneration period, ξ_i is the number of customers arrived during the i-th regeneration period. Assume that $a = E\xi_i < \infty$, $\tau = E\tau_i < \infty$, and $\lambda = \lim\limits_{t\to\infty} \frac{X(t)}{t} = a\tau^{-1}$ *a.s.*.

Let $\{v_n\}_{n=1}^{\infty}$ be the sequence of independent identical distributed random variables and it does not depend on the input flow and service times. The random variable v_n can be an improper one, i.e. $\alpha = P\{v_n = \infty\} \geq 0$. Denote $C(x) = P\{v_n \leq x | v_n < \infty\}$. Moreover v_n bounds the waiting time of the nth customer in the system, i.e. if the nth customer does not start it's service during the time v_n then it leaves the system without service at all. Let $q(t)$ be a number of customers in the system at time t. Under some additional assumptions $q(t)$ is a regenerative process and θ_i is it's point of regeneration if $q(\theta_i - 0) = 0$.

Theorem 1. The process $q(t)$ is ergodic iff $\rho = \alpha\lambda\beta^{-1} < 1$.

The proof is based on the lemma about stochastic boundedness and ergodicity of the regenerative process proved in [Afanasyeva, Tkachenko, 2014] and construction of majorizing process. Then results for regenerative process with finite mean of the period of regeneration [Thorisson, 1987] are applied.

First we give the following result concerning so called super-heavy traffic situation ($\rho \geq 1$).

Theorem 2. If $\rho > 1$ ($\rho = 1$) and for some $\delta > 0$

$$E\tau_1^{2+\delta} < \infty,\ E\xi_1^{2+\delta} < \infty,\ E(\eta_1^i)^{2+\delta} < \infty,\ i = \overline{1,r}, \qquad (\star)$$

then the normalized process $\hat{q}_n(t) = \frac{q(nt)-\beta(\rho-1)nt}{\hat{\sigma}\sqrt{n}}$ weakly converges on any finite interval $[0,t]$ to Brownian motion (absolute value of Brownian motion) as $n \to \infty$. Here

$$\hat{\sigma}^2 = \sigma_X^2 + \sigma_\beta^2,\ \sigma_X = \frac{\alpha\sigma_\xi^2}{\tau} + \frac{(\alpha a)^2\sigma_\tau^2}{\tau^3} - \frac{2a\alpha^2 cov(\xi_1,\tau_1)}{\tau^2},$$

$$\sigma_\beta^2 = \sum_{i=1}^{r} \sigma_i^2\beta_i^3,\ \sigma_\tau^2 = Var(\tau_1),\ \sigma_\xi^2 = Var(\xi_1),\ \sigma_i^2 = Var(\eta_1^i),\ i = \overline{1,r}.$$

In order to prove this theorem we use Brownian approximation for modified multichannel systems [Iglehart, Whitt, 1970] and construct two majorizing systems.

Second we focus on the process $q(t)$ in the heavy-traffic situation ($\rho \uparrow 1$). We consider time-compression asymptotic. Namely the input flow is given by the relation

$$X_n(t) = X\left(\rho^{-1}\left(1 - \frac{1}{\sqrt{n}}\right)t\right)$$

so that the traffic coefficient depends on the parameter n and $\rho_n \uparrow 1$ as $n \to \infty$. Let $q_n(t)$ be the process $q(t)$ for the system with input flow $X_n(t)$.

Theorem 3. Under conditions ($\star$) the normalized process $\tilde{q}_n(t) = \frac{q_n(nt)}{\sqrt{n}}$ weakly converges on any finite interval $[0,t]$ as $n \to \infty$ to the diffusion process with reflecting at the origin and coefficients $(-\beta, \tilde{\sigma}^2)$, where $\tilde{\sigma}^2 = \sigma_\beta^2 + \frac{\sigma_X^2}{\rho}$.

The proof is based on the construction of the functional limit of the fluid process [Whitt, 2002] and some estimates for number of customers in the system.

References

1. Afanasyeva, L. and Tkachenko, A., Multichannel Queueing Systems with Regenerative Input Flow // Theory of Probability and Its Applications, 2014, V.58, No. 2, P. 174–192.
2. Thorisson, H., A complete coupling proof of Blackwell's renewal theorem// Stochastic Processes and Their Applications, 1987, V. 26, P. 87–97.
3. Iglehart, D.L. and Whitt, W., Multiple Channel Queues in Heavy Traffic. I // Advances in Applied Probability, 1970, V. 2, No. 1, P. 150–177.
4. Whitt W. Stochastic-process limits: an introduction to stochastic-process limits and their application to queues: Springer Science and Business Media, 2002.

Waiting-time tail probabilities in queue with regenerative input flow and unreliable server

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We consider a single-server queueing system with a regenerative input flow $A(t)$ ($Reg/G/1$). Here $A(t)$ is the number of customers arrived during $[0,t]$. The random variable θ_i is said to be the ith regeneration moment of $A(t)$ and $\tau_i = \theta_i - \theta_{i-1}$ is the ith regeneration period. Let $\xi_i = A(\theta_i - 0) - A(\theta_{i-1})$ be the number of arrived customers during the ith regeneration period. Assume that $\mathsf{E}\xi_i < \infty$ and $\mathsf{E}\tau_i < \infty$. The *intensity* of $A(t)$ is the limit $\lambda = \lim\limits_{t\to\infty} \frac{A(t)}{t}$ with probability one (w.p.1). It is easy to see that $\lambda = \frac{\mathsf{E}\xi_1}{\mathsf{E}\tau_1}$.

Assumption 1. *The greatest common divisor of numbers* ($i = 1, 2, \dots$) *such that* $\mathsf{P}(\xi_1 = i) > 0$ *is equal to one.*

Service times of customers are defined by the sequence $\{\eta_n\}_{n=1}^{\infty}$ that consists of i.i.d. random variables and does not depend on $A(t)$. The distribution function of η_1 is $B(x)$, $b = \mathsf{E}\eta < \infty$ and $b(s) = \mathsf{E}e^{-s\eta_1}$.

Let $W(t)$ be the virtual waiting time process and $W_n = W(\theta_n - 0)$, $w_n = W(t_n - 0)$. Here t_n is the moment of the nth customer arrival at the system. Define functions $\Psi(x) = \lim\limits_{t\to\infty} \mathsf{P}(W(t) \le x)$, $\Phi(x) = \lim\limits_{n\to\infty} \mathsf{P}(W_n \le x)$ and $F(x) = \lim\limits_{n\to\infty} \mathsf{P}(w_n \le x)$.

It is known (see e.g. [1]) that $\Psi(x)$, $\Phi(x)$ and $F(x)$ are distribution functions if and only if the *traffic intensity*

$$\rho = \lambda b < 1.$$

Here we aim to analyze the asymptotic behavior of functions $\Psi(x)$, $\Phi(x)$ and $F(x)$ as $x \to \infty$. For any distribution function $F(x)$ we put $\bar{F}(x) = 1 - F(x)$. As usual $f(x) \sim h(x)$ as $x \to \infty$ if $\lim\limits_{x\to\infty} \frac{f(x)}{h(x)} = 1$. As in [3], we define the following class of distributions.

Definition 1. *A distribution function $F(x)$ on $\mathbb{R}$ with finite mean belongs to the class of the strong subexponential distributions if*

$$\int_0^x \bar{F}(x-y)\bar{F}(y)dy \sim 2m\bar{F}(x),$$

where $m = \int_0^\infty \bar{F}(y)dy$.

Theorem 1. *Let $B(x)$ be a strong subexponential distribution function and Assumption 1 be fulfilled.*

(i) If there exists $c > b$ such that $\mathsf{P}(\xi > x/c) = o(\bar{B}(x))$, then

$$\bar{\Phi}(x) \sim \frac{\lambda}{1-\lambda b}\int_x^\infty \bar{B}(y)dy \quad \text{as } x \to \infty. \tag{1}$$

(ii) If there exists $c > b$ such that $\sqrt{\mathsf{P}(\xi > x/c)} = o(\bar{B}(x))$, then (1) *holds for the function $\bar{F}(x)$.*

(iii) If there exists $c > b$ such that $\sqrt{\mathsf{P}(\xi > x/c)} = o(\bar{B}(x))$, $\mathsf{E}\tau^2 < \infty$, then (1) *holds for the function $\bar{\Psi}(x)$.*

Further we consider a queueing system $Reg/G/1$ with an unreliable server. The breakdowns of the server occur only when it is occupied by a customer. Besides, if the server is in the working state then breakdowns appear at random in the sense that the time until the next breakdown is exponentially distributed with a parameter ν. After breakdown the server is repaired during the random time with distribution function $D(x)$, mean d and $d(s) = \int\limits_0^\infty e^{-sx}dD(x)$. There are various disciplines for continuation of the service after server restoration. Here we consider the *preemptive repeat different* service discipline when service is repeated from the start and the service time after restoration is independent of the origin service time. This discipline was considered in the pioneering paper [4] where the notion of *completion time* was introduced. This notion made it possible to apply results for systems without interruptions to investigate a system with unreliable server. Let us remind that completion time is the sojourn time of the customer on the server with regard of repairs of the server (if there are). Introduce the distribution function of completion time $B_c(x)$ and mean b_c, then

$$\bar{B}_c(x) \sim \frac{1-b(\nu)}{b(\nu)}\bar{D}(x) \quad \text{as } x \to \infty,$$

$$b_c = \frac{1-b(\nu)}{b(\nu)}\left(\frac{1}{\nu}+d\right).$$

Corollary 1. *For a queueing system with an unreliable server let the distribution function of the repair time $D(x)$ be strong subexponential. All conditions from (iii) of Theorem 1 are satisfied with $B_c(x)$ instead of $B(x)$ and Assumption 1 holds. Then*

$$\bar{\Phi}(x) \sim \bar{\Psi}(x) \sim \bar{F}(x) \sim \frac{\lambda(1-b(\nu))}{(1-\lambda b_c)b(\nu)}\int_x^\infty \bar{D}(y)dy \quad \textit{as } x \to \infty.$$

As we can see from Corollary 1, if we have the preemptive repeat different service discipline then the distribution function of service time has no influence on asymptotic behavior of $\bar{\Psi}(x)$. A queueing system $M/G/1$ with *preemptive resume* service discipline was considered in [2]. For this discipline the customer's service after a restoration continuous from the point at which it was interrupted. It was shown that if $B(x)$ and $D(x)$ are regularly varying distributions then

$$\bar{B}_c(x) \sim \bar{B}\left(\frac{x}{1+\nu d}\right) + \nu b\bar{D}(x) \quad \text{as } x \to \infty.$$

Thus if the function $\bar{D}(x)$ is lighter than $\bar{B}(x)$ as $x \to \infty$ then the distribution $B(x)$ defines asymptotics of $\bar{\Psi}(x)$.

These results mean that the asymptotic behavior of $\bar{\Psi}(x)$ as $x \to \infty$ is completely defined by the intensity λ of the input flow and the distribution function of the service time (or repair time if the server is unreliable). Therefore the structure of the input flow does not play any role if condition (iii) holds. This condition means that the tail of ξ is essentially lighter than tail of η. We strongly believe that otherwise the dominate part may belong to the distribution of ξ.

References

1. Afanaseva L.G., Bashtova E.E. Coupling method for asymptotic analysis of queues with regenerative input and unreliable server // Queueing Systems. 2014. V. 76, № 2. P. 125–147.
2. Aibatov S.Z. Large deviation probabilities for the system $M/G/1/\infty$ with an unreliable server // Teor. Ver. Prim. 2016. V. 61, № 1. P. 1–9. (in Russian)
3. Foss S., Korshunov D., Zachary S. An introduction to heavy-tailed and subexponential distributions. New York: Springer, 2011.
4. Gaver D.P. A waiting line with interrupted service including priority // J. Rl. Stat. Soc. B. 1962. V. 24, P. 73–90.

Limit theorems for queuing system with an infinite number of servers and regenerative input flow

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This paper focuses on a queuing system S with an infinite number of servers and regenerative input flow $X(t)$, given on $(\Omega, \mathcal{F}, \mathsf{P})$. All trajectories are left-continuous non-decreasing functions with integer values, $X(0) = 0$. The definition of this process is [3].

Definition. *The flow $X(t)$ is regenerative if there exists an increasing sequence of random variables $\{\theta_i\}_{i\geq 0}$, $\theta_0 = 0$, such that the sequence*

$$\{\kappa_i\}_{i=0}^{\infty} = \{(X(\theta_{i-1}+t) - X(\theta_{i-1})), \theta_i - \theta_{i-1}, t \in [0; \theta_i - \theta_{i-1})\}_{i=0}^{\infty}$$

consists of independent identically distributed random elements on $(\Omega, \mathcal{F}, \mathsf{P})$.

The value θ_i is called the i-th moment of regeneration, $\tau_i = \theta_i - \theta_{i-1}$ is the i-th period of regeneration.

We assume that $\{\tau_i\}_{i=1}^{\infty}$ are independent identically distributed random variables(i.i.d.r.v.), with distribution function $F(x)$.

Let $\xi_i = X(\theta_i) - X(\theta_{i-1})$ be the number of customers arrived during the i-th regeneration period.

Service times of customers $\{\eta_{ij}, j = 1, \ldots, \xi_i, i \geq 1\}$ are i.i.d.r.v. with distribution function $B(t)$. Denote $\overline{B}(t) = 1 - B(t)$. We assume that the following condition is fulfilled.

Condition. *For the function $\overline{B}(t)$ asymptotic behavior takes place*

$$\overline{B}(t) \sim \frac{\mathcal{L}(t)}{t^{\beta}} \ \textit{as } t \to \infty,$$

$0 < \beta < 1$. *Here $\mathcal{L}(t)$ is slowly varying function as $t \to \infty$[4].*

The focus of this paper is the process $q(t)$, which is the number of customers in the system S at time t.

Denote $\lambda = \frac{1}{1-\beta}\frac{E\xi_1}{E\tau_1}$. Let us formulate our results.

Theorem 1.*Suppose that $E\tau_1^r < \infty$, $r > 2$, $E\xi_1^2 < \infty$. Then*

$$\frac{q(t) - \lambda t^{1-\beta}\mathcal{L}(t)}{\sqrt{t^{1-\beta}\mathcal{L}(t)}} \xrightarrow{d} \mathcal{N}\left(0, \lambda\right), \ \textit{as } t \to \infty.$$

Theorem 2.*Suppose that $E\tau_1^r < \infty$, $r > 2$, $E\xi_1^2 < \infty$. Then*

$$\frac{q(t)}{t^{1-\beta}\mathcal{L}(t)} \xrightarrow{p} \lambda, \ \textit{as } t \to \infty.$$

Description of auxiliary systems and their relationship with the initial. To study the asymptotic behavior of the queue length in the system S we introduce two auxiliary systems. In the first system S_1 customers enter only at the beginning of the regeneration period $[\theta_{i-1}, \theta_i]$, by group ξ_i , $i \geq 1$. In the second system group of customers comes at the end of the period of regeneration, we denote it S_2. Let $q_i(t)$ be the number of customers in the system S_i at time t respectively, $i = 1, 2$.

Let $\Delta(t)$ be the number of customers that left system S_1, but not left S_2 at time t. For $\Delta(\theta_n)$ the following representation holds

$$\Delta(\theta_n) = \sum_{i=1}^{n}\sum_{j=0}^{\xi_i} \kappa_{ij}(\theta_n),$$

where $\kappa_{ij}(\theta_n) = \begin{cases} 1, & \theta_n - \theta_i \leq \eta_{ij} < \theta_n - \theta_{i-1}, \\ 0, & \text{otherwise.} \end{cases}$

Note that $q_1(t)$, $q_2(t)$, and $q(t)$ satisfy following relations with probability 1

$$q_1\left(\theta_{N(t)}\right) \leq q(t) \leq q_2\left(\theta_{N(t)}\right) + \xi_{N(t)+1}, \ q_2\left(\theta_{N(t)}\right) = q_1\left(\theta_{N(t)}\right) + \Delta\left(\theta_{N(t)}\right).$$

The last inequality shows that in order to obtain limit theorems for the process $q(t)$, we need limit theorems for $q_1\left(\theta_{N(t)}\right)$, $\xi_{N(t)+1}$, $\Delta_{N(t)}$.

Some auxiliary results. Let us formulate the limit theorem for $q_1\left(\theta_{N(t)}\right)$, $\xi_{N(t)+1}$, $\Delta\left(\theta_{N(t)}\right)$. Denote $E_n = \frac{1}{1-\beta}\frac{E\xi_1}{E\tau_1^\beta}n^{1-\beta}\mathcal{L}(n)$.

Theorem 3. *Let $E\tau_1^r < \infty$, $r > 2$, $E\xi_1^2 < \infty$. Then*

$$\frac{q_1(\theta_n) - E_n}{\sqrt{E_n}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Lemma 1. *Let $E\xi_1 < \infty$, $E\tau_1^r < \infty$, $r > 2$. Then*

$$\frac{\Delta(\theta_n)}{n^{\frac{1-\beta}{2}}} \xrightarrow{p} 0, \text{ as } n \to \infty.$$

Lemma 2. *For any $k \in \mathbb{N}$*

$$\lim_{t\to\infty} P\left(\xi_{\tau_{N(t)+1}} = k\right) = \frac{1}{E\tau_1}\int_0^\infty P\left(\tau_2 > x, \xi_2 = k\right)dx.$$

In order to obtain similar results for $q_1(\theta_{N(n)})$, and $\Delta(\theta_{N(n)})$ as in Theorem 3 and Lemma 1, we need the following Theorem.

Theorem 4. (Theorem 5,[2]) *Let $Y_n \xrightarrow{d} Y$, $n \to \infty$ and*

1. $\frac{N_n}{n} \xrightarrow{p} N$, $n \to \infty$ *with* $\mathsf{P}(0 < N < \infty) = 1$,
2. *If Y_n is R-mixing with respect to $\sigma(N)$, that is, for each A such that* $\mathsf{P}(N \in A) > 0$ *holds* $\mathsf{P}(Y_n \in *|N \in A) \to \mathsf{P}(Y \in *)$,
3. *if* $\Delta_{n,c} = \max\limits_{|m-n|<nc} |Y_m - Y_n|$, *then*

$$\lim_{c\to 0}\sup_{\{B:\mathsf{P}(B)>0\}}\limsup_{n\to\infty}\mathsf{P}(\Delta_{n,c} > \varepsilon|N \in B) = 0.$$

Then $Y_{N_n} \xrightarrow{d} Y$.

Verification conditions of this theorem is based on results of following lemmas. Let $\mathcal{F}$ be σ-algebra, formed by variables $\{\xi_i, \theta_i\}_{i=1}^\infty$.

Lemma 3. *Sequence $\{Z_m = Y_m - Y_n\}_{m\geq n}$ forms a conditional N-demimartingale given $\mathcal{F}$ [1].*

Lemma 4. *Let $E\tau_1^r < \infty$, $r > 2$, $E\xi_1^2 < \infty$. For any constant $0 < c < 1$ there exists $n_0(c)$ such that for $n > n_0(c)$ we have the following inequality for limits in probability*

1. $$\lim_{n\to\infty} \frac{E\left(\left(Y_n - Y_{n(1-c)}\right)^2 | \mathcal{F}\right)}{n^{1-\beta}\mathcal{L}(n)} \le C_1\left(1 - c^{1-\beta} - (1-c)^{1-\beta}\right) + C_2 c^{1-\beta},$$

2. $$\lim_{n\to\infty} \frac{E\left(\left(Y_n - Y_{n(1+c)}\right)^2 | \mathcal{F}\right)}{n^{1-\beta}\mathcal{L}(n)} \le C_1\left((1+c)^{1-\beta} - c^{1-\beta} - 1\right) + C_2 c^{1-\beta}.$$

for some C_1, C_2

References

1. Rao B. L. S. P. Associated sequences, demimartingales and nonparametric inference. - Springer Science & Business Media, 2012.
2. Durrett R. T., Resnick S. I. Weak convergence with random indices //Stochastic Processes and their Applications. - 1977. - T. 5. - No. 3. - P. 213-220.
3. Cox D. R. et al. Renewal theory. - London : Methuen, 1962. - T. 1.
4. Seneta E. Regularly varying functions. - 1976.

On ergodic averaging with and without invariant measure*

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The classical Birkhoff ergodic theorem in its most popular version says that the time average along a single typical realization of a Markov process is equal to the space average with respect to the ergodic invariant distribution. This result is one of the cornerstones of the entire ergodic theory and its numerous applications. In this talk I'll address two questions related to this subject: how large is the set of typical realizations, in particular when there are no invariant distributions, and how this is connected to properties of the so called natural measures (limits of images of "good" measures under the action of the system).

Our main results concern with necessary and sufficient conditions under which for a given reference measure (e.g. Lebesgue measure), whose support might be much larger than the support of the invariant one, the set of typical initial points is of full measure. It turns out that one

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of the main assumptions here is the ergodicity of the natural measure. To deal with the situation when the invariant measure does not exist we extend the notion of ergodicity to measures being non invariant.

To give an example of a system without invariant distributions satisfying our setup, consider the following deterministic Markov process: a family of maps from the unit disc $X := \{(\phi, R) : \ 0 \le \phi < 2\pi, \ 0 \le R \le 1\}$ into itself defined in the polar coordinates (ϕ, R) by the relation:

$$T(\phi, R) :=$$

$$:= \begin{cases} (\phi + 2\pi\alpha + \beta(R - r) \bmod 2\pi, \ \gamma(R - r) + r) & \text{if } r(R - r) \neq 0 \\ (\phi + 2\pi\alpha \bmod 2\pi, \ (1 + r)/2) & \text{otherwise} \end{cases}$$

with the parameters $\alpha, \beta, \gamma, r \ \in \ (0, 1)$. One can show that for any probability measure μ absolutely continuous with respect to the Lebesgue measure, the sequence of measures $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \mu$ (Cesaro averages of images of the measure μ under the action of T) converges weakly to a certain limit measure μ_T on the circle $\{R = r\}$, but this measure is no longer invariant. Depending on the choice of the rotation parameters $\alpha, \beta \in (0, 1)$ properties of the set of μ_T-typical points turn out to be very different. In particular, if these parameters are rationally independent, the limit measure is unique and the set of μ_T-typical points coincides with the entire unit disk.

Questions discussed above turn out to be especially actual in the case of large systems, when even in the presence of ergodic invariant measures, their supports cover only a small part of the phase space.

Asymptotic behavior and stability of some applied probability models

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Applied probability research domains such as insurance, inventory and dams, finance, queueing theory, reliability and some others can be considered as special cases of decision making under uncertainty (or risk management) aimed at the systems performance optimization, thus eliminating or minimizing risk.

The crucial question in all investigations pertaining to decision making is: How to choose an appropriate mathematical model? There always exists a trade-off between simplicity and precision. A simple model gives a possibility of easily obtaining an explicit solution. However the poor model fit is the fist source of decision errors. A complicated model giving precise description may also lead to errors. Namely, numerical solution needed for complicated models and parameters variability constitute the second source of decision errors. Perturbations of the underlying processes provide the third source of decision errors. Thus, the model stability to small fluctuations of model parameters and distributions of basic processes is a must, see, e.g., [1–3] and references therein.

It is well known that the same mathematical model can arise in various applications. So, for certainty, we are going to speak below about insurance models, although many conclusions will be valid for other fields. The primary task of insurer is redistribution of risks and satisfaction of policyholders claims. This explains the popularity of reliability approach, that is, thorough analysis of ruin probability. The classical Cramér-Lundberg model introduced in 1903 and significantly developed during the first part of the 20-th century is still the base for many investigations and generalizations. Being a corporation, insurance company has a secondary but very important task, namely, dividends payment to its shareholders. So, the alternative so-called cost approach was started by De Finetti in 1957, see [4]. Modern period in actuarial sciences evolution is characterized by consideration of a larger class of stochastic processes. Not only compound Poisson processes describe insurance company performance but renewal and regeneration processes, martingales, diffusion, Markov, semi-Markov and Lévy processes. Moreover, interplay between insurance and finance is typical nowadays, see, e.g. [5]. Banks are selling insurance and reinsurance contracts whereas insurance companies are interested in investment and capital injections, see, e.g. [6–8].

Since decisions about reinsurance and dividends payment are usually made at the end of the year discrete-time models were introduced, see, e.g. [9–12]. It turned out that such models can also be used for approximation of continuous-time ones.

We begin by treating the models studied in [11,12] and their generalization to the case of two-dimensional claims. Optimal and asymptotically optimal policies are established solving Bellman functional equations. Systems stability is verified by means of Sobol' method and local sensitivity analysis. The results are used to implement a numerical algorithm letting obtain some approximations to optimal solutions for continuous-time models. Convergence rate to limit distribution is also studied using various metrics, see, e.g. [13]. In case of unknown distributions of underlying processes it is appropriate to use stochastic orders to compare various models. Finally, we apply empirical processes (see [14]) to get statistical inference enabling us to use a sequence of observations for calculations of optimal policy parameters.

References

1. Oakley J.E. and O'Hagan A. Probabilistic sensitivity analysis of complex models: a Bayesian approach // J. R. Statist. Soc. B. 2004. V. 66, Part 3. P. 751–769.
2. Saltelli A., Tarantola S. and Campolongo F. Sensitivity analysis as an ingredient of modeling // Statist. Sci. 2000. V. 15. P. 377–395.
3. Sobol' I.M. Sensitivity analysis for nonlinear mathematical models // Math. Modlng Comput. Expt. 1993. V. 1. P. 407–414.
4. De Finetti B. Su un'impostazione alternativa della teoria collettiva del rischio // Transactions of the XV-th International Congress of Actuaries. 1957. V. 2. P. 433–443.
5. Yang H., Gao Wei and Li J. Asymptotic ruin probabilities for a discrete-time risk model with dependent insurance and financial risks // Scandinavian Actuarial Journal. 2016. № 1. P. 1–17.
6. Dickson D.C.M. and Waters H.R. Some optimal dividends problems // ASTIN Bulletin. 2004. V. 34. P. 49–74.
7. Eisenberg J., Schmidli H. Optimal control of capital injections by reinsurance in a diffusion approximation // Blätter der DGVFM. 2009. V. 30, № 1. P. 1–13.
8. Kulenko N., Schmidli H. Optimal dividend strategies in a Cramér-Lundberg model with capital injections // Insurance: Mathematics

and Economics. 2008. V. 43. P. 270–278.
9. Bulinskaya E. On the cost approach in insurance // Review of Applied and Industrial Mathematics. 2003. V. 10, № 2. P. 276–286.
10. Li Sh., Lu Yi and Garrido J. A review of discrete-time risk models // Rev. R.Acad. Cien. Serie A. Mat. 2009. V. 103, № 2. P. 321–337.
11. Bulinskaya E. Asymptotic Analysis of Insurance Models with Bank Loans // New Perspectives on Stochastic Modeling and Data Analysis. Athens, Greece: ISAST, 2014. P. 255–270.
12. Bulinskaya E. and Gromov A. Asymptotic Behavior of the Processes Describing Some Insurance Models // Communications in Statistics - Theory and Methods. 2016. V. 45, № 6. P. 1778–1793.
13. Rachev S.T., Stoyanov S.V. and Fabozzi F.J. Advanced Stochastic Models, Risk Assessment, Portfolio Optimization. Hoboken, New Jersey: J.Wiley and Sons. 2008.
14. Shorack G.R., Wellner J.A. Empirical Processes with Application to Statistics. New York: J.Wiley and Sons. 1986.

Stability of the solution in the optimal reinsurance problem

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We consider a periodic - review insurance model under the following assumptions. One-period insurance claims form a sequence of independent identically distributed nonnegative random variables $\{X_k\}, k \geq 1$. Each X_k has a distribution as that of the random variable X with finite mean and cumulative distribution function F_X.
In order to avoid ruin the insurer maintains the company surplus above a chosen level a by capital injections at the end of each period. A nonproportional reinsurance is applied for minimization of total expected discounted injections $h_{n_X}(u)$ during a given planning horizon of n periods, where u is the initial surplus of the insurance company, $u \geq a$. Insurance and reinsurance premiums are calculated using the expected value principle. The optimal reinsurance strategy for this problem has been established in the paper[1].

This work relies on the results obtained in [1] and considers the stability of minimal expected injections to the fluctuation of claim distribution. More precisely, suppose one-period claim $X_k, k \geq 1$ has the same distribution as random variable Y with cumulative distribution function F_Y, which in turn differs from function F_X. In this case, how does the amount of optimal capital injections change? The following theorem gives us the answer to this question under the assumption that random variables X and Y are close in Kantorovich metric. The metric is calculated according to the definition in [2] and equals to $\kappa(X,Y) = \int_0^\infty |F_X(t) - F_Y(t)|dt$.

Theorem. Let X and Y be nonnegative random variables with finite mean defined on the same probability space, than the following inequality holds for every $n \geq 1$

$$\sup_{u \geq a} |h_{n_X}(u) - h_{n_Y}(u)| \leq \left(\sum_{i=0}^{n-1} \alpha^i C_{n-i} \right) (1 + l + m)\kappa(X,Y),$$

where $0 < \alpha < 1$ is a discount coefficient, $l > 1$ and $m > l$ denote safety loadings on the insurance and reinsurance premiums respectively, h_{n_Y} refers to minimal discounted expected injections when one-period claim distribution function is equal to F_Y, $C_{n-i} = \frac{1-\alpha^{n-i}}{1-\alpha}$.

Due to the fact that in practice theoretical distributions are usually unknown, we also investigate the stability of the solution, when distribution functions F_X, F_Y are replaced by their empirical estimates.

References

1. Bulinskaya E.V., Gusak J.V, Muromskaya A.A. Discrete-time Insurance Model with Capital Injections and Reinsurance. Methodology and Computing in Applied Probability. 2015. V.17, 4, P. 899–914.
2. Rachev S.T., Klebanov L., Stoyanov S.V., Fabozzi F. The Methods of Distances in the Theory of Probability and Statistics. Springer-Verlag New York, 2013.

On a classical risk model with a step barrier dividend strategy

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We consider an insurance company performance with dividends payment. According to the Cramer-Lundberg model, the surplus of the insurance company paying dividends is as follows:

$$X(t) = x + ct - S(t) - D(t), \ t \geq 0.$$

Here $\{S(t)\}$ is a compound Poisson process with intensity λ, $D(t)$ denotes total dividends paid until t and $x = X(0)$. Premiums are acquired continuously at the rate c and the claim amounts are nonnegative i.i.d. random variables with distribution function $F(y)$. Let T also denote the time of ruin, namely, $T = \inf\{t : \ X(t) < 0\}$.

Dividends are paid in conformity with some dividend strategy. One of the most popular dividend strategies are so-called constant barrier strategies. In the framework of the constant barrier strategy with level b, no dividends are paid whenever $X(t) < b$ and dividends at the rate c are paid whenever $X(t) = b$. If $X(t) > b$, an amount $X(t) - b$ is paid out immediately as dividends. Constant barrier strategies were considered in many papers devoted to dividend theory, such as Gerber *et al.* [1] and Buhlmann [2]. However constant barrier strategies have one significant disadvantage, namely, the barrier level can not be changed throughout the life of the insurance company. In this regard we examine modified barrier strategies, according to which the barrier level b can be changed after the moments of claim occurrences T_i (step barrier strategies).

At first let us consider the model with the barrier that can be changed only a finite number of times (after each of the first $(n-1)$ claim occurrences). In this case barrier level changes up to the ruin time in conformity with the following rule: $b = b_i$ on the interval $[T_{i-1}, T_i)$ for $1 \leq i \leq n-1$ (we assume $T_0 = 0$) and $b = b_n$ if $t \geq T_{n-1}$. The step barrier function is supposed to be nondecreasing: $b_1 \leq b_2 \leq \ldots \leq b_n$.

Let $V(x, b)$ and $V(x, b_1, \ldots, b_n)$ denote the expected discounted dividends paid until ruin in the models with constant barrier and step barrier strategies respectively. Then the following theorem holds true.

Theorem 1. For all $0 \leq x \leq b_1$ and $b_1 \leq b_2 \leq \ldots \leq b_n, \ n \geq 2$, the function $V(x, b_1, \ldots, b_n)$ can be expressed as:

$$V(x, b_1, \ldots, b_n) = V(x, b_n) + \sum_{i=1}^{n-1} [1 - V'(b_i, b_n)] \, V_{[T_{i-1}, T_i)}(x, b_1, \ldots, b_i),$$

where $V_{[T_{i-1}, T_i)}(x, b_1, \ldots, b_i)$ is the mathematical expectation of the discounted dividends paid on $[T_{i-1}, T_i)$.

Remark. Functions $V_{[T_{k-1}, T_k)}(x, b_1, \ldots, b_k), \ 1 \leq k \leq n-1$, can be calculated sequentially with the help of the law of total probability:

$$V_{[0,T_1)}(x, b_1) = \frac{c}{\lambda+\delta} e^{-(\lambda+\delta)\frac{b_1-x}{c}},$$

$$V_{[T_{k-1},T_k)}(x, b_1, \dots, b_k) =$$
$$= \int_0^{\frac{b_1-x}{c}} \lambda e^{-(\lambda+\delta)t} \int_0^{x+ct} V_{[T_{k-2},T_{k-1})}(x+ct-y, b_2, \dots, b_k) dF(y)dt+$$
$$+ \int_{\frac{b_1-x}{c}}^{\infty} \lambda e^{-(\lambda+\delta)t} \int_0^{b_1} V_{[T_{k-2},T_{k-1})}(b_1-y, b_2, \dots, b_k) dF(y)dt, \quad k \geq 2.$$

Now let us consider the probability of ruin $\psi(x) = P(T < \infty | X(0) = x)$ in the model with a barrier level that can be changed after every claim occurrence $T_j, j \geq 1$, (i.e. infinite number of times). It is also assumed that the equation

$$\lambda + rc = \lambda \int_0^{\infty} e^{ry} dF(y)$$

has the unique positive solution R. If this solution exists we call it the adjustment coefficient or the Lundberg exponent ([3], [4]). The coefficient R plays an important role in the estimation of the ruin probabilities, in particular, in our model we have the following result.

Theorem 2. The ruin probability $\psi(x)$ satisfies the inequality

$$\psi(x) \leq e^{-Rx} + \frac{Rc}{\lambda} \sum_{i=1}^{\infty} e^{-Rb_i}.$$

This theorem is a generalization of the Lundberg inequality which is proved for the classical risk model without dividend payments.

Examples of the step barrier functions, for which the upper bound for the ruin probability is less than 1, will be given. Note that in the framework of the constant barrier dividend strategy the ruin of the insurance company occurs almost surely.

References

1. Gerber H.U., Shiu E.S.W. and Smith N. Maximizing dividends without bankruptcy // ASTIN Bulletin. 2006. V. 1, № 36. P. 5–23.
2. Buhlmann H. Mathematical methods in risk theory. Berlin, Heidelberg: Springer-Verlag, 1970.
3. Bulinskaya E.V. Risk theory and reinsurance, part 2 (in Russian). Moscow: Moscow State University, 2006.

4. Schmidli H. Stochastic control in insurance. London: Springer-Verlag, 2008.

Weakly supercritical branching walks with heavy tails*

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Branching random walks (BRWs) are usually described in terms of birth, death and walk of particles. We consider a continuous-time symmetric BRW on a multidimensional lattice. In [1], a detailed description of such BRW for the case of finite variance of jumps and one source of branching is given. In the present work we study the case of BRWs with heavy tails when intensities of the underlying random walk are subjected to a condition leading to infinite variance of jumps, see, e.g., [2].

Quite a number of authors investigated the random walks with heavy tails, see the bibliography in [3]. Most of them, as a rule, have restricted themselves to consideration of the one-dimensional case. In the multidimensional case of a spatially homogeneous symmetric random walk with infinite variance of jumps, proofs of global limit theorems for the transition probabilities of a random walk, in the case when the temporal and spatial variables jointly tend to infinity, can be found in [4]. The corresponding results were proved under an additional regularity condition imposed on the transition intensities of a random walk. In [5], a multidimensional analog of the well-known Watson's lemma (see, e.g., [6]) was proven which helps to investigate in [5] an asymptotic behaviour of the transition probabilities for fixed spatial coordinates without making any additional assumptions on the transition intensities.

The goal of the work is to apply obtained results to find the asymptotic behavior of the moments for BRWs with infinite variance of jumps and the only branching source. Employing the scheme suggested in [1] for BRWs with a finite variance of jumps, we find the generating functions, differential and integral equations for the moments of the numbers of particles, as in an arbitrary lattice point as on the entire lattice for BRWs with infinite variance of jumps. Abandonment of the finiteness of the variance of jumps, as was shown in [2,7], leads to changes

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in the BRW's properties: as a result the BRW becomes transient even on one- and two-dimensional lattices. The minimal value of the intensity of the branching source, under which in the spectrum of the operator describing the evolution of the mean numbers of particles there appear a positive eigenvalue, is called critical. The asymptotic behaviour of Green's functions and eigenvalues of the evolutionary operator, for the BRW with heavy tails and intensities of the source exceeding but still close the critical value, is studied in [8]. Notice that their behaviour differs drastically from the case of finite variance of jumps. Using the results of [8] we obtain a number of statements on asymptotic behavior of the first moments of the numbers of particles for weakly supercritical BRWs. The obtained results are generalized then to the case of a finite number of branching sources for weakly supercritical BRWs with heavy tails.

References

1. Yarovaya E.B. Branching Random Walks in a Non-homogeneous Environment. Moscow: Tsentr Prikladnykh Issledovanii pri Mekhaniko-Matematicheskom Fakul'tete MGU, 2007. [in Russian]
2. Yarovaya E. Branching Random Walks with Heavy Tails // Communications in Statistics - Theory and Methods. 2013. No. 42:16. P. 2301–2310.
3. Borovkov A., Borovkov K. Asymptotic Analysis of Random Walks. Heavy-Tailed Distributions. Cambridge: Cambridge University Press, 2008.
4. Agbor A.,Molchanov S., Vainberg B. Global limit theorems on the convergence of multidimensional random walks to stable processes //Stochastics and Dynamics. 2015. No. 15:3. 1550024.
5. Rytova A.I., Yarovaya E.B. Multidimensional Watson Lemma and Its Applications // Mathematical Notes. 2016. Vol. 99, No. 3. P. 64–70.
6. Fedoryuk M.V. Asymptotics: Integrals and Series, in Mathematical Reference Library. Moscow: Nauka, 1987. [in Russian]
7. Yarovaya E. Criteria for Transient Behavior of Symmetric Branching Random Walks on $\mathbf{Z}$ and $\mathbf{Z}^2$ // New Perspectives on Stochastic Modeling and Data Analysis. Athens: ISAST, 2014. P. 283–294,
8. Yarovaya E.B. The Structure of the Positive Discrete Spectrum of the Evolution Operator Arising in Branching Random Walks // Doklady Mathematics. 2015. Vol. 92, No. 1. P. 1–4.

Asymptotic behaviour of generalized renewal processes and some applications

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Let $\{T_n\}_{n\geq 1}$ be a sequence of independent non-negative random variables, F_j is the distribution function of variable T_{ql+j} for some fixed integer $l > 1$, $q = 1, 2, \ldots$. Let $\{X_i\}_{i=0,\ldots,k-1}$ be another sequence of independent random variables (r.v.), each X_i has its distribution function G_i. The sequences $\{T_n\}$ and $\{X_i\}$ are also supposed to be independent.

Let us define generalized delayed periodical renewal process in the following way: $S_n = X_0 + \ldots + X_n$, $0 \leq n \leq k-1$, whereas $S_n = S_{k-1} + T_1 + \ldots + T_{n-k+1}$ for $n \geq k$.

The partial sums S_n are called the renewals and the summands T_i and X_i are the intervals between the renewals.

The main object of our consideration is the counting process

$$N_t = \min\{k \geq 0 : S_k > t\},$$

representing the number of renewals that have occurred by time t.

The purpose of the talk is investigation of the asymptotic behaviour of defined renewal process N_t and application of obtained results to the risk theory.

Using tauberian theorem (see, e.g. [1]) the asymptotic form of renewal function is found. The results concerning simple renewal processes are also used (see, e.g. [2], [3], [4]). The analogues of the strong law of large numbers, central limit theorem and functional limit theorem are proved.

The main steps of research:

1. Finding the limit behaviour and distribution of the process on the basis of asymptotic behaviour of sequence of renewals.

2. Introduction of the auxiliary random elements by means of centering and normalization of partial sums of process.

3. Proof of the weak convergence of auxiliary elements to a Wiener process.

4. At last we proceed to the process constructed according to counting process N_t using the theorem about the random change of measure (see, e.g. [5]).

Theorem 1. Let S_n be a generalized delayed periodical renewal process and N_t a counting process associated with it. Suppose that all the summands T_{ql+i} have finite mathematical expectation $\mu_i < \infty$, $i = 1, \ldots, l$. Then with probability 1

$$\frac{N_t}{t} \to \frac{l}{\mu},$$

where $\mu = \mu_1 + \ldots + \mu_l$.

Theorem 2. Suppose that r.v.'s T_{ql+i} have finite mathematical expectations $\mu_i < \infty$ and variances $0 < \sigma_i^2 < \infty$, respectively, $i = 1, \ldots, l$, and r.v.'s X_j have finite mathematical expectations ν_j. Then, as $t \to \infty$, we have

$$\frac{N_t - tl\mu^{-1}}{\sigma l \sqrt{t\mu^{-3}}} \xrightarrow{d} \xi \sim N(0,1),$$

where $\mu = \mu_1 + \ldots + \mu_l$, $\sigma^2 = \sigma_1^2 + \ldots + \sigma_l^2$, $\xrightarrow{d}$ denotes weak convergence, and $\xi \sim N(0,1)$ means that r.v. ξ has standard Gaussian distribution.

Theorem 3. Let us define the sequence of random functions

$$Z_n(t,\omega) = \frac{N_{nt}(\omega) - ntl\mu^{-1}}{\sigma l \mu^{-3/2}\sqrt{n}},$$

where $\mu = \mu_1 + \ldots + \mu_l$, $\sigma^2 = \sigma_1^2 + \ldots + \sigma_l^2$.

For defined random functions $Z_n(t,\omega)$ the following expression holds:

$$Z_n \xrightarrow{D} W,$$

where W is a Wiener process and $\xrightarrow{D}$ denotes weak convergence in the space $D[0,1]$.

References

1. Feller W. An Introduction to Probability Theory and its Applications, Vol. 2. New York: Wiley, 1971.
2. Afanasyeva L., Bulinskaya E. Stochastic Processes in Queueing Theory and Inventory Control. Moscow: Moscow State University Press, 1980. [In Russian.]
3. Borovkov A. Probability Theory. Moscow: Editorial URSS, 1999. [In Russian.]
4. Cox D.R. Renewal Theory. Methuen and Company, Ltd. 1962.
5. Billingsley P. Convergence of Probability Measures. New York: Wiley, 1968.

Branching random walks. Spectral approach*

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Stochastic processes with generation and transport of particles are used in different areas of nature sciences: statistical physics, chemical kinetics, etc. [1-2]. Behavior of processes with generation and transport of particles in many ways determined by properties of a particle motion and a dimension of the space in which the particles evolve. In [3] for studying a change of homopolymers spatial structure under the influence of temperature there was suggested an approach based on a resolvent analysis of the evolutionary operator. Unlike to [3] we consider a multidimensional integer lattice instead of $\mathbf{R}^d$ and a random walk instead of a Brownian motion [4]. The description of a random walk in terms of Green's function allows us to offer a general approach to investigation of random walks with finite as well as with infinite variance of jump.

We consider a continuous-time symmetric branching random walk on a multidimensional lattice with a finite set of the particle generation centres, i.e. branching sources [5]. Branching random walks models are relevant in numerous applications, including population studies. Particular attention is paid to branching random walks with infinite variance jumps. Such branching random walks can be used in modeling of complex stochastic systems with singular spacial dynamics, implying the existence of heavy-tailed distributions of random walk jumps [6].

The main object of study is the evolutionary operator for the mean number of particles both at an arbitrary point and on the entire lattice. The existence of positive eigenvalues in the spectrum of an evolutionary operator results in the exponential growth of the number of particles in branching random walks, called supercritical in such case. For supercritical branching random walks, it is shown that the amount of positive eigenvalues of the evolutionary operator, counting their multiplicity, does not exceed the amount of branching sources on the lattice, while the maximal of these eigenvalues is always simple [6]. We demonstrate that the appearance of multiple lower eigenvalues in the spectrum of the evolutionary operator can be caused by a kind

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of 'symmetry' in the spatial configuration of branching sources [5]. The presented results are based on Green's function representation of transition probabilities of an underlying random walk and cover not only the case of the finite variance of jumps but also a less studied case of infinite variance of jumps.

References

1. Gärtner J., Molchanov S. Parabolic problems for the Anderson model. I. Intermittency and related topics // Comm. Math. Phys. 1990. No. 132:3 P. 613–655.
2. Gärtner J., Molchanov S. Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. // Probab. Theory Related Fields. 1998. No. 111:1, P. 7–55.
3. Cranston M., Koralov L., Molchanov S., and B. Vainberg B. Continuous model for homopolymers // J. Funct. Anal. (2009), No. 256:82, P. 656–2696.
4. Molchanov S., Yarovaya E. //Proceedings of the Steklov Institute of Mathematics. 2013. Vol. 282, P. 186–201.
5. Yarovaya E.B. Positive Discrete Spectrum of the Evolutionary Operator of Supercritical Branching Walks with Heavy Tails // Methodology and Computing in Applied Probability. First online: 12 March, 2016. P. 1–17.
6. Yarovaya E.B. The Structure of the Positive Discrete Spectrum of the Evolution Operator Arising in Branching Random Walks // Doklady Mathematics. 2015. Vol. 92, No. 1. P. 1–4.

Asymptotic properties of marginal distributions in a polling system with batch renewal inputs and limited service policy*

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Consider a polling system with $m < \infty$ stations, batch renewal inputs, limited service policy, and fixed switch-over times. Inter-arrival

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times at the j-th station are i.i.d. non-negative random variables with probability density function $a_j(t)$. A batch is size b with probability $f_j(b)$, $b = 1, 2, \ldots$. Server sojourn time at the j-th station is a non-random constant $T_{2j-1} > 0$. During this interval at most ℓ_j customers at the station can be serviced. These may be both the customers present at the station at the beginning of the service slot and the newly arriving customers. Service times of individual customers are not specified and are mutually dependent in a way they manage to leave before the slot ends. Serviced customers leave the queueing system. After station $j < m$ the server switches to the next station $(j+1)$, after the station $j = m$, the station 1 is visited. Switch-over time is a non-random constant $T_{2j} > 0$.

We observe the queueing system at epochs τ_i, $i = 0, 1, \ldots$ of service periods and switch-over periods termination. Denote by $\Gamma_i \in \Gamma$, $\Gamma = \{\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(2m)}\}$ the server state during time interval $(\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots$, by $\Gamma_0 \in \Gamma$ the server state at time τ_0, where $\Gamma^{(2j-1)}$ stands for service at the station j and $\Gamma^{(2j)}$ stands for switch-over from the station j to the station $j+1$ if $j < m$ and from station m to station 1 if $j = m$. Let $\kappa_{j,i}$ be the queue length at the station j at time τ_i, $\zeta_{j,i}$ be the residual inter-arrival time at time τ_i at the station j, $i = 0, 1, \ldots$. Put $\kappa_i = (\kappa_{1,i}, \ldots, \kappa_{m,i})$, $\zeta_i = (\zeta_{1,i}, \ldots, \zeta_{m,i})$. In [1] a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ was constructed and a stochastic sequence
$$\{(\Gamma_i, \kappa_i, \zeta_i); i = 0, 1, \ldots\} \tag{1}$$
was defined on it and the Markov property was proven for sequence (1) and for sequences
$$\{(\Gamma_i, \kappa_{j,i}, \zeta_{j,i}); i = 0, 1, \ldots\}, \qquad j = 1, \ldots, m. \tag{2}$$
Stochastic sequences (1) and (2) are general Markov chains [2] with uncountable state spaces. Further, given that for each $j = 1, \ldots, m$ there exists a $t_j^0 > 0$ such that $a_j(t) = 0$ for $t < t_j^0$ and $a_j(t) > 0$ for $t \geqslant t_j^0$ and $f_j(1) > 0$, in [1] the general Markov chain (1) was proven to be ψ-irreducible [2]. Moreover, if each $a_j(t)$ is continuous for $t > t_j^0$ then some small sets [2] of the general Markov chain (1) are known.

Denote by $Q_{j,i}(r, x, y) = \mathsf{P}(\{\omega\colon \Gamma_i = \Gamma^{(r)}, \kappa_{j,i} = x, \zeta_{j,i} < y\})$, $j = 1, \ldots, m$ marginal probability distributions for Markov chains (2) and by
$$\Psi_{j,i}(z, s, r) = \sum_{x=0}^{\infty} \int_0^{\infty} z^x e^{-sy}\, d_y Q_{j,i}(r, x, y)$$
their integral transforms. Set $\lambda_j = \int\limits_0^{\infty} t a_j(t)\, dt$ and $\bar{\lambda}_j = \lambda_j \sum\limits_{b=1}^{\infty} b\, f_j(b)$.

We claim the following.

Theorem 1. Let series $\hat{f}_j(z) = \sum_{b=1}^{\infty} b\, f_j(b)$ and functions $\Psi_{j,0}(z, s, r)$, $s \geqslant 0$, $r = 1, 2, \ldots, 2m$ be analytic in an open disk $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$ and some $j = 1, \ldots, m$. Further, let

$$\bar{\lambda}_j(T_1 + \ldots + T_{2m}) - \ell_j < 0.$$

Then the functions $\Psi_{j,i}(z, s, r)$, $s \geqslant 0$, $r = 1, 2, \ldots, 2m$, and $i = 0, 1, \ldots$ are uniformly bounded w.r.t. z in an open disk $|z| \leqslant 1 + \varepsilon_1$, $0 < \varepsilon_1 < \varepsilon$, and the sequence $\{\mathsf{E}\kappa_{j,i}; i = 0, 1, \ldots\}$ is bounded.

Theorem 1 plays an elemental role in establishing a sufficient condition for the existence of a stationary probability distribution for the Markov chain (1) by iterative-dominating approach.

References

1. Zorine A.V. A cybernetic model of cyclic control of conflicting flows with an after-effect // Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki. 2014. V. 156, № 3. P. 66–75 (in Russian).
2. Meyn S.P., Tweedie R.L. Markov chains and stochastic stability. London: Springer-Verlag, 1993.

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